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Unified Einstein-Virasoro Master Equation in the General Non-Linear Sigma Model

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ABSTRACT

The Virasoro master equation (VME) describes the general affine-Virasoro construction $T = L^{ab} J_a J_b + i D^a \partial J_a$ in the operator algebra of the WZW model, where L^{ab} is the inverse inertia tensor and D^a is the improvement vector. In this paper, we generalize this construction to find the general (one-loop) Virasoro construction in the operator algebra of the general non-linear sigma model. The result is a unified Einstein-Virasoro master equation which couples the spacetime spin-two field L^{ab} to the background fields of the sigma model. For a particular solution L_G^{ab} , the unified system reduces to the canonical stress tensors and conventional Einstein equations of the sigma model, and the system reduces to the general affine-Virasoro construction and the VME when the sigma model is taken to be the WZW action. More generally, the unified system describes a space of conformal field theories which is presumably much larger than the sum of the general affine-Virasoro construction and the sigma model with its canonical stress tensors. We also discuss a number of algebraic and geometrical properties of the system, including its relation to an unsolved problem in the theory of G -structures on manifolds with torsion.

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1 Introduction

There have been two successful approaches to the construction of conformal field theories,

- The general affine-Virasoro construction [1–7]
- The general non-linear sigma model [8–13] (1.1)

but, although both approaches have been formulated as Einstein-like systems [12, 2], the relation between the two has remained unclear.

In the general affine-Virasoro construction, a large class of exact Virasoro operators [1, 3]

$$T = L^{ab} {}_* J_a J_b {}_* + i D^a \partial J_a \quad (1.2a)$$

$$a, b = 1 \dots \dim(g) \quad (1.2b)$$

are constructed as quadratic forms in the currents J of the general affine Lie algebra [14, 15]. The coefficients $L^{ab} = L^{ba}$ and D^a are called the inverse inertia tensor and the improvement vector respectively. The general construction is summarized [1, 3] by the (improved) Virasoro master equation (VME) for L and D , and this approach is the basis of irrational conformal field theory [7] which includes the affine-Sugawara [15–18] and coset constructions [15, 16, 19] as a small subspace. The construction (1.2) can also be considered as the general Virasoro construction in the operator algebra of the WZW model [20, 21], which is the field-theoretic realization of the affine algebras. See Ref [7] for a more detailed history of affine Lie algebra and the affine-Virasoro constructions.

In the general non-linear sigma model, a large class of Virasoro operators [22]

$$T = -\frac{1}{2\alpha'} G_{ij} \partial x^i \partial x^j + \mathcal{O}(\alpha'^0) = -\frac{1}{2\alpha'} G^{ab} \Pi_a \Pi_b + \mathcal{O}(\alpha'^0) \quad (1.3a)$$

$$G^{ab} = e_i^a G^{ij} e_j^b, \quad \Pi_a = G_{ab} e_i^b \partial x^i \quad (1.3b)$$

$$i, j, a, b = 1, \dots, \dim(M) \quad (1.3c)$$

is constructed in a semiclassical expansion on an arbitrary manifold M , where G_{ij} is the metric on M and G^{ab} is the inverse of the tangent space metric. These are the canonical or conventional stress tensors of the sigma model and this construction is summarized [12, 22] by the Einstein equations of the sigma model, which couple the metric G , the antisymmetric tensor field B and the dilaton Φ . In what follows we refer to these equations as the conventional Einstein equations of the sigma model, to distinguish them from the generalized Einstein equations obtained below.

In this paper, we unify these two approaches, using the fact that the WZW action is a special case of the general sigma model. More precisely, we study the general Virasoro construction

$$T = -\frac{1}{\alpha'} L_{ij} \partial x^i \partial x^j + \mathcal{O}(\alpha'^0) = -\frac{1}{\alpha'} L^{ab} \Pi_a \Pi_b + \mathcal{O}(\alpha'^0) \quad (1.4a)$$

$$i, j, a, b = 1, \dots, \dim(M) \quad (1.4b)$$

at one loop in the operator algebra of the general sigma model, where L is a symmetric second-rank spacetime tensor field, the inverse inertia tensor, which is to be determined. The unified construction is described by a system of equations which we call

- the Einstein-Virasoro master equation

of the general sigma model. This system, which is summarized in Section 4, describes the covariant coupling of the spacetime fields L , G , B and Φ_a , where the vector field Φ_a generalizes the derivative $\nabla_a \Phi$ of the dilaton Φ .

The unified system contains as special cases the two constructions in (1.1): For the particular solution

$$L^{ab} = L_G^{ab} = \frac{G^{ab}}{2} + \mathcal{O}(\alpha'), \quad \Phi_a = \Phi_a^G = \nabla_a \Phi \quad (1.5)$$

the general stress tensors (1.4) reduce to the conventional stress tensors (1.3) and the Einstein-Virasoro master equation reduces to the conventional Einstein equations of the sigma model. Moreover, the unified system reduces to the general affine-Virasoro construction and the VME when the sigma model is taken to be the WZW action. In this case we find that the contribution of Φ_a to the unified system is precisely the known improvement term of the VME.

More generally, the unified system describes a space of conformal field theories which is presumably much larger than the sum of the general affine-Virasoro construction and the sigma model with its canonical stress tensors.

The system exhibits a number of interesting algebraic and geometric properties, including K -conjugation covariance [15, 16, 19, 1] and an important connection with the theory of G -structures [23] on manifolds with torsion. By comparison with the known exact form of the VME, we are also able to point to a number of candidates for exact relations in the general sigma model.

2 Background

To settle notation and fix concepts which will be important below, we begin with a brief review of the two known constructions,

- The general affine-Virasoro construction
- The general non-linear sigma model.

2.1 The general Affine-Virasoro Construction

The improved VME

The general affine-Virasoro construction begins with the currents of a general affine Lie algebra [14, 15]

$$J_a(z)J_b(w) = \frac{G_{ab}}{(z-w)^2} + \frac{if_{ab}^c J_c(w)}{z-w} + \text{reg.} \quad (2.1)$$

where $a, b = 1 \dots \dim g$ and f_{ab}^c are the structure constants of g . For simple g , the central term in (2.1) has the form $G_{ab} = k\eta_{ab}$ where η_{ab} is the Killing metric of g and k is the level of the affine algebra. Then the general affine-Virasoro construction is [1]

$$T = L^{ab} {}_* J_a J_b {}_* + iD^a \partial J_a \quad (2.2)$$

where the coefficients $L^{ab} = L^{ba}$ and D^a are the inverse inertia tensor and the improvement vector respectively. The stress tensor T is a Virasoro operator

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{reg.} \quad (2.3)$$

iff the improved VME [1]

$$L^{ab} = 2L^{ac}G_{cd}L^{db} - L^{cd}L^{ef}f_{ce}^a f_{df}^b - L^{cd}f_{ce}^f f_{df}^{(a} L^{b)e} - f_{cd}^{(a} L^{b)c} D^d \quad (2.4a)$$

$$D^a(2G_{ab}L^{be} + f_{ab}^d L^{bc} f_{cd}^e) = D^e \quad (2.4b)$$

$$c = 2G_{ab}(L^{ab} + 6D^a D^b) \quad (2.4c)$$

is satisfied* by L and D , and the central charge of the construction is given in (2.4c). The unimproved VME [1, 3] is obtained by setting the improvement vector D to zero.

The improved VME has been identified [2] as an Einstein-Maxwell system with torsion on the group manifold, with left-invariant inverse metric

$$G^{ij} = e_a^i L^{ab} e_b^j \quad (2.5a)$$

$$D_i L^{ab} = 0 \quad (2.5b)$$

where e_a^i is the inverse left-invariant vielbein and L^{ab} is the inverse of the tangent space metric.

Another property that will be useful below is that the improved VME is covariant under automorphisms of g

$$(L')^{ab} = L^{cd} \rho_c^a \rho_d^b, \quad (D')^a = D^b \rho_b^a \quad (2.6a)$$

*Our convention is $A^{(a} B^{b)} = A^a B^b + A^b B^a$, $A^{[a} B^{b]} = A^a B^b - A^b B^a$.

$$c(L', D') = c(L, D), \quad \rho \in \text{Aut}(g) \quad (2.6b)$$

$$\rho_a^c \rho_b^d G_{cd} = G_{ab}, \quad \rho_a^c \rho_b^d f_{cd}^e = f_{ab}^c \Omega_c^e \quad (2.6c)$$

with invariant central charge.

K-conjugation invariance

A central property of the VME at zero improvement[†] is K -conjugation covariance [15, 16, 19, 1] which says that all solutions come in K -conjugate pairs L and \tilde{L} ,

$$L^{ab} + \tilde{L}^{ab} = L_g^{ab}. \quad (2.7a)$$

$$T + \tilde{T} = T_g \quad (2.7b)$$

$$c + \tilde{c} = c_g \quad (2.7c)$$

$$T(z)\tilde{T}(w) = \text{reg.} \quad (2.7d)$$

whose K -conjugate stress tensors T, \tilde{T} commute and add to the affine-Sugawara construction [15–18] on g

$$T_g = L_g^{ab} *_J J_a J_b *_J. \quad (2.8)$$

For simple g , the inverse inertia tensor and the central charge of the affine-Sugawara construction are

$$L_g^{ab} = \frac{\eta^{ab}}{2k + Q_g} = \frac{\eta^{ab}}{2k} + \mathcal{O}(k^{-2}) = \frac{G^{ab}}{2} + \mathcal{O}(k^{-2}) \quad (2.9a)$$

$$c_g = \frac{\dim(g)}{1 + \frac{Q_g}{2k}} = \dim(g) \left(1 - \frac{Q_g}{2k}\right) + \mathcal{O}(k^{-2}) \quad (2.9b)$$

$$Q_g \eta_{ab} = -f_{ac}^d f_{bd}^c \quad (2.9c)$$

where η^{ab} is the inverse Killing metric of g and Q_g is the quadratic Casimir of the adjoint. K -conjugation covariance can be used to generate new solutions $\tilde{L} = L_g - L$ from old solutions L and the simplest application of the covariance generates the coset constructions [15, 16, 19] as $\tilde{L} = L_g - L_h = L_{g/h}$.

Semiclassical expansion

At zero improvement, the high-level or semiclassical expansion [24, 7] of the VME has been studied in some detail. On simple g , the leading term in the expansion has the form

$$L^{ab} = \frac{P^{ab}}{2k} + \mathcal{O}(k^{-2}), \quad c = \text{rank}(P) + \mathcal{O}(k^{-1}) \quad (2.10a)$$

$$P^{ac} \eta_{cd} P^{db} = P^{ab} \quad (2.10b)$$

[†] K -conjugation in the presence of the improvement term is discussed in Section 5.9

where P is the high-level projector of the L theory. These are the solutions of the classical limit of the VME,

$$L^{ab} = 2L^{ac}G_{cd}L^{db} + \mathcal{O}(k^{-2}) \quad (2.11)$$

but a semiclassical quantization condition [24] provides a restriction on the allowed projectors. In the partial classification of the space of solutions by graph theory [5, 25, 7], the projectors P are closely related to the adjacency matrices of the graphs.

Irrational conformal field theory

Given also a set of antiholomorphic currents \bar{J}_a , $a = 1 \dots \dim(g)$, there is a corresponding antiholomorphic Virasoro construction

$$\bar{T} = L^{ab} \bar{J}_a \bar{J}_b + iD^a \partial \bar{J}_a \quad (2.12)$$

with $\bar{c} = c$. Each pair of stress tensors T and \bar{T} then defines a conformal field theory (CFT) labelled by L and D . Starting from the modules of affine $g \times g$, the Hilbert space of a particular CFT is obtained [26, 27, 7] by modding out by the local symmetry of the Hamiltonian.

It is known that the CFTs of the master equation have generically irrational central charge, even when attention is restricted to the space of unitary theories, and the study of all the CFTs of the master equation is called irrational conformal field theory (ICFT), which contains the affine-Sugawara and coset constructions as a small subspace.

In ICFT at zero improvement, world-sheet actions are known for the following cases: the affine-Sugawara constructions (WZW models [20, 21]), the coset constructions (spin-one gauged WZW models [28]) and the generic ICFT (spin-two gauged WZW models [26, 29, 30]). The spin-two gauge symmetry of the generic ICFT is a consequence of K -conjugation covariance.

See Ref [7] for a comprehensive review of ICFT, and Ref [31] for a recent construction of a set of semiclassical blocks and correlators in ICFT.

WZW model

The stress tensors (2.2) and (2.12) of the general affine-Virasoro construction are general quadratic forms in the currents of affine $g \times g$. As such, these stress tensors may also be considered as the general Virasoro construction in the operator algebra of the WZW model on the group manifold G , where g is the algebra of G .

For simple G , the Minkowski space WZW action is [20, 21]

$$S_{WZW} = -\frac{k}{8\pi\chi} \int dt d\sigma \text{Tr}(g^{-1} \partial_+ g g^{-1} \partial_- g) - \frac{k}{12\pi\chi} \int \text{Tr}(g^{-1} dg)^3 \quad (2.13a)$$

$$g(T) \in G, \quad \text{Tr}(T_a T_b) = \chi \eta_{ab} \quad (2.13b)$$

$$\partial_{\pm} = \partial_t \pm \partial_{\sigma} \quad (2.13c)$$

where $g(T)$ is the group element in irrep T of g , η_{ab} is the Killing metric of g and k is the level of the affine algebra. The group element g is dimensionless, and we may introduce dimensionless coordinates y^i , $i = 1 \dots \dim(g)$, such that

$$-ig^{-1}\frac{\partial}{\partial y^i}g = e_i^a T_a, \quad -ig\frac{\partial}{\partial y^i}g^{-1} = \bar{e}_i^a T_a \quad (2.14a)$$

$$gT_ag^{-1} = \Omega_a^b T_b, \quad \bar{e}_i^a = -e_i^b \Omega_b^a \quad (2.14b)$$

$$\Omega_a^c \Omega_d^b G_{cd} = G_{ab}, \quad \Omega_a^c \Omega_b^d f_{cd}^e = f_{ab}^c \Omega_c^e \quad (2.14c)$$

$$\frac{\partial}{\partial y^i} \Omega_a^b + e_i^d f_{da}^c \Omega_c^b = 0 \quad (2.14d)$$

where e and \bar{e} are respectively the left and right invariant vielbein on G and Ω is the adjoint action of g .

Continuing in this direction, one rewrites the WZW action as a special case of the general non-linear sigma model, a subject to which we will return in the following section.

2.2 The general non-linear sigma model

The general non-linear sigma model has been extensively studied in [32, 33, 8, 34, 9, 10, 35, 36, 11, 12, 37, 38, 22, 13].

The Euclidean action of the general non-linear sigma model is

$$S = \frac{1}{2\alpha'} \int d^2z (G_{ij} + B_{ij}) \partial x^i \bar{\partial} x^j \quad (2.15a)$$

$$d^2z = \frac{dx dy}{\pi}, \quad z = x + iy \quad (2.15b)$$

$$H_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}. \quad (2.15c)$$

Here x^i , $i = 1 \dots \dim(M)$ are coordinates with the dimension of length on a general manifold M and α' , with dimension length squared, is the string tension or Regge slope. The fields G_{ij} and B_{ij} are the (covariantly constant) metric and antisymmetric tensor field on M .

We also introduce a covariantly constant vielbein e_i^a , $a = 1 \dots \dim(M)$ on M and use it to translate between Einstein and tangent-space indices, e.g.

$$G_{ij} = e_i^a G_{ab} e_j^b \quad (2.16)$$

where G_{ab} is the covariantly constant metric on tangent space. Covariant derivatives are defined as

$$\nabla_i v_a = \partial_i v_a - \omega_{ia}^b v_b, \quad \nabla_i v^a = \partial_i v^a + v^b \omega_{ib}^a \quad (2.17a)$$

$$\nabla_i G_{ab} = \partial_i G_{ab} = \nabla_i e_j^a = 0 \quad (2.17b)$$

$$[\nabla_i, \nabla_j] v^a = R_{ijb}^a v^b \quad (2.17c)$$

$$R_{ija}{}^b = (\partial_i \omega_j - \partial_j \omega_i - [\omega_i, \omega_j])_a{}^b \quad (2.17d)$$

$$\nabla_a v_b = e_a{}^i \nabla_i v_b = \partial_a v_b - \omega_{ab}{}^c v_c \quad (2.17e)$$

where ω is the spin connection, $R_{ija}{}^b$ is the Riemann tensor and $R_{ab} = R_{acb}{}^c$ is the Ricci tensor. It will also be convenient to define the generalized connections and covariant derivatives with torsion,

$$\hat{\nabla}_i^\pm v_a = \partial_i v_a - \hat{\omega}_{ia}^\pm{}^b v_b \quad (2.18a)$$

$$\hat{\omega}_{ia}^\pm{}^b = \omega_{ia}{}^b \pm \frac{1}{2} H_{ia}{}^b \quad (2.18b)$$

$$\hat{\nabla}_i^\pm G_{ab} = \hat{\nabla}_j^\pm e_i{}^a = 0 \quad (2.18c)$$

$$\hat{R}_{ija}{}^b = (\partial_i \hat{\omega}_j^\pm - \partial_j \hat{\omega}_i^\pm - [\hat{\omega}_i^\pm, \hat{\omega}_j^\pm])_a{}^b \quad (2.18d)$$

$$[\hat{\nabla}_a^\pm, \hat{\nabla}_b^\pm] v^c = \mp H_{ab}{}^d \hat{\nabla}_d^\pm v^c + \hat{R}_{abd}{}^c v^d \quad (2.18e)$$

$$R_{cdab}^\pm = R_{cdab} \pm \frac{1}{2} (\nabla_c H_{dab} - \nabla_d H_{cab}) + \frac{1}{4} (H_{ca}{}^f H_{fdb} - H_{da}{}^f H_{fcb}) \quad (2.18f)$$

$$\hat{R}^\pm = \hat{R}_a{}^\pm{}^a = R + \frac{1}{4} H^2, \quad (H^2)_{ij} = H_i{}^{kl} H_{jkl} \quad (2.18g)$$

where $\hat{\omega}_{iab}^\pm$ is antisymmetric under (a, b) interchange and \hat{R}_{ijab}^\pm is pairwise antisymmetric in (i, j) and (a, b) .

Following Banks, Nemeschansky and Sen [22], the canonical or conventional stress tensors of the general sigma model have the form

$$T_G = -\frac{G_{ij}}{2\alpha'} \partial x^i \partial x^j + \partial^2 \Phi + T_1 + \mathcal{O}(\alpha') \quad (2.19a)$$

$$= -\frac{G^{ab}}{2\alpha'} \Pi_a \Pi_b + \partial^2 \Phi + T_1 + \mathcal{O}(\alpha') \quad (2.19b)$$

$$\bar{T}_G = -\frac{G_{ij}}{2\alpha'} \bar{\partial} x^i \bar{\partial} x^j + \bar{\partial}^2 \Phi + \bar{T}_1 + \mathcal{O}(\alpha') \quad (2.19c)$$

$$= -\frac{G^{ab}}{2\alpha'} \bar{\Pi}_a \bar{\Pi}_b + \bar{\partial}^2 \Phi + \bar{T}_1 + \mathcal{O}(\alpha') \quad (2.19d)$$

$$\Pi_a = G_{ab} e_i{}^b \partial x^i, \quad \bar{\Pi}_a = G_{ab} e_i{}^b \bar{\partial} x^i \quad (2.19e)$$

where Φ is the dilaton and T_1, \bar{T}_1 are finite one-loop counterterms which depend on the renormalization scheme. The condition that T_G and \bar{T}_G are one-loop conformal is [12]

$$R_{ij} + \frac{1}{4} (H^2)_{ij} - 2 \nabla_i \nabla_j \Phi = \mathcal{O}(\alpha') \quad (2.20a)$$

$$\nabla^k H_{kij} - 2 \nabla^k \Phi H_{kij} = \mathcal{O}(\alpha') \quad (2.20b)$$

$$c_G = \bar{c}_G = \dim(M) + 3\alpha' (4|\nabla \Phi|^2 - 4\nabla^2 \Phi + R + \frac{1}{12} H^2) + \mathcal{O}(\alpha'^2) \quad (2.20c)$$

where (2.20a) and (2.20b) are the conventional Einstein equations of the sigma model and (2.20c) is the central charge of the construction. The result for the central charge

includes two-loop information, but covariant constancy of the field-dependent part of the central charge follows by Bianchi identities from the Einstein equations, so all three relations in (2.20) can be obtained with a little thought from the one-loop calculation. It will also be useful to note that the conventional Einstein equations (2.20a),(2.20b) can be written in either of two equivalent forms

$$\hat{R}_{ij}^{\pm} - 2\hat{\nabla}_i^{\pm}\hat{\nabla}_j^{\pm}\Phi = \mathcal{O}(\alpha') \quad (2.21)$$

by using the generalized quantities (2.18) with torsion.

WZW data

The WZW action (2.13a) is a special case of the general sigma model (2.15a) on a group manifold G . Identifying the vielbein e on M with the left-invariant vielbein e on G in (2.14a), we find for WZW that

$$x^i = \sqrt{\alpha'} y^i, \quad G_{ab} = k\eta_{ab}, \quad H_{ab}{}^c = \frac{1}{\sqrt{\alpha'}} f_{ab}{}^c \quad (2.22)$$

where y^i , $i = 1 \dots \dim(g)$ are the dimensionless coordinates introduced for WZW in (2.14), $f_{ab}{}^c$ and η_{ab} are the structure constants and the Killing metric of g and k is the level of the affine algebra. From this data, one computes

$$\omega_{ab}{}^c = -\frac{1}{2\sqrt{\alpha'}} f_{ab}{}^c \quad (2.23a)$$

$$R_{abc}{}^d = -\frac{1}{4\alpha'} f_{ab}{}^e f_{ec}{}^d, \quad R_{ab} = -\frac{Q_g}{4\alpha'} \eta_{ab}, \quad R = -\frac{Q_g}{4\alpha' k} \dim(g) \quad (2.23b)$$

$$(H^2)_{ab} = \frac{Q_g}{\alpha'} \eta_{ab}, \quad H^2 = \frac{kQ_g \dim(g)}{\alpha'} \quad (2.23c)$$

where Q_g is the quadratic Casimir of the adjoint. Finally, one finds

$$\hat{\omega}_{ab}^{+c} = 0, \quad \hat{\omega}_{ab}^{-c} = -\frac{1}{\sqrt{\alpha'}} f_{ab}{}^c \quad (2.24a)$$

$$\hat{R}_{ija}^{\pm b} = 0 \quad (2.24b)$$

for the generalized quantities (2.18) with torsion. Manifolds with vanishing generalized Riemann tensors are called parallelizable [35, 37].

The relations between the classical WZW currents J, \bar{J} and the quantities $\Pi, \bar{\Pi}$ in (2.19e) are

$$J_a = ik\eta_{ab} e_i{}^b \partial y^i = \frac{i}{\sqrt{\alpha'}} \Pi_a \quad (2.25a)$$

$$\bar{J}_a = ik\eta_{ab} \bar{e}_i{}^b \bar{\partial} y^i = \frac{-i}{\sqrt{\alpha'}} (\Omega^{-1})_a{}^b \bar{\Pi}_b \quad (2.25b)$$

where e and \bar{e} are the left and right invariant vielbeins in (2.14a) and Ω is the adjoint action of g . It follows that the classical limits of the affine-Sugawara stress tensors

$$T_G \rightarrow T_g = -\frac{L_g^{ab}}{\alpha'} \Pi_a \Pi_b + \mathcal{O}(\alpha'^0) = L_g^{ab} J_a J_b + \mathcal{O}(\alpha'^0) \quad (2.26a)$$

$$\bar{T}_G \rightarrow \bar{T}_g = -\frac{L_g^{ab}}{\alpha'} \bar{\Pi}_a \bar{\Pi}_b + \mathcal{O}(\alpha'^0) = L_g^{ab} \bar{J}_a \bar{J}_b + \mathcal{O}(\alpha'^0) \quad (2.26b)$$

$$L_g^{ab} \approx \frac{G^{ab}}{2} = \frac{\eta^{ab}}{2k} \quad (2.26c)$$

are correctly obtained as a special case of the conventional stress tensors (2.19). The result in (2.26b) follows because the adjoint action is an inner automorphism of g .

We can also check at the WZW level that the affine-Sugawara stress tensors (2.26) are one-loop conformal. Using (2.20), (2.21) and (2.24b), we find that the Einstein equations are satisfied for the affine-Sugawara constructions with $\nabla\Phi = 0$ and central charge

$$c_G = \dim(g) - \frac{\alpha'}{2} H^2 + \mathcal{O}(\alpha'^2) \quad (2.27a)$$

$$= \dim(g) \left(1 - \frac{Q_g}{2k}\right) + \mathcal{O}(k^{-2}) \quad (2.27b)$$

which is in agreement with the exact form of the affine-Sugawara central charge c_g in (2.9b).

2.3 Strategy

It is unlikely that the semiclassical form of the Einstein-Maxwell formulation [2] of the VME can be identified directly with the conventional Einstein equations (2.20) of the sigma model on a group manifold. One reason is that the semiclassical inverse metrics of the Einstein-Maxwell formulation

$$G^{ij} = \frac{1}{2k} e_a^i P^{ab} e_b^j + \mathcal{O}(k^{-2}) \quad (2.28)$$

are singular because P is a projector (see eqs (2.5) and (2.10)). On the other hand, we know for example that the coset constructions are contained in both the VME and the canonical stress tensors of the sigma model, so the possibility remains open that the Einstein-Maxwell form of the VME is in some sense dual to the Einstein equations of the sigma model (see also the related remarks in Section 7).

Our strategy here is a straightforward generalization of the VME to the sigma model, following the relation of the general affine-Virasoro construction to the WZW model. In the operator algebra of the general sigma model, we use the technique of Banks et al. [22] to study the general Virasoro construction

$$T = -\frac{L_{ij}}{\alpha'} \partial x^i \partial x^j + \mathcal{O}(\alpha'^0) = -\frac{L^{ab}}{\alpha'} \Pi_a \Pi_b + \mathcal{O}(\alpha'^0) \quad (2.29a)$$

$$\bar{T} = -\frac{\bar{L}_{ij}}{\alpha'} \bar{\partial} x^i \bar{\partial} x^j + \mathcal{O}(\alpha'^0) = -\frac{\bar{L}^{ab}}{\alpha'} \bar{\Pi}_a \bar{\Pi}_b + \mathcal{O}(\alpha'^0) \quad (2.29b)$$

$$\bar{\partial} T = \partial \bar{T} = 0 \quad (2.29c)$$

$$\langle T(z)T(w) \rangle = \frac{c/2}{(z-w)^4} + \frac{2\langle T(w) \rangle}{(z-w)^2} + \frac{\langle \partial T(w) \rangle}{(z-w)} + \text{reg.} \quad (2.29d)$$

$$\langle \bar{T}(\bar{z})\bar{T}(\bar{w}) \rangle = \frac{\bar{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2\langle \bar{T}(\bar{w}) \rangle}{(\bar{z}-\bar{w})^2} + \frac{\langle \bar{\partial} \bar{T}(\bar{w}) \rangle}{(\bar{z}-\bar{w})} + \text{reg.} \quad (2.29e)$$

where the dilatonic contribution is included at $\mathcal{O}(\alpha'^0)$ and L is a symmetric second-rank spacetime tensor field (the inverse inertia tensor) to be determined.

According to eq. (2.19), the stress tensors in (2.29) reduce, at leading order in α' , to the conventional stress tensors T_G , \bar{T}_G of the general sigma model when the inverse inertia tensors are taken as

$$L = \bar{L} = L_G \quad (2.30a)$$

$$L_G^{ab} = \frac{G^{ab}}{2} + \mathcal{O}(\alpha') \quad (2.30b)$$

where G^{ab} is the inverse tangent space metric of the sigma model action (2.15a). In this sense, the conventional stress tensors of the sigma model generalize the affine-Sugawara stress tensors (2.8) of the WZW model (see eq. (2.9a)), and the general sigma model stress tensors (2.29a), (2.29b) generalize the stress tensors (2.2) of the general affine-Virasoro construction.

3 Classical preview of the construction

The classical limit of the general construction (2.29a), (2.29b) can be studied with the classical equations of motion of the general sigma model, which can be written in two equivalent forms

$$\bar{\partial} \Pi_a + \bar{\Pi}_b \Pi_c \hat{\omega}^{+bc}_a = \partial \bar{\Pi}_a + \Pi_b \bar{\Pi}_c \hat{\omega}^{-bc}_a = 0 \quad (3.1)$$

where $\Pi, \bar{\Pi}$ are defined in (2.19e) and $\hat{\omega}^\pm$ are the generalized connections (2.18b) with torsion.

One then finds that the classical stress tensors are holomorphic and antiholomorphic respectively

$$T = -\frac{L^{ab}}{\alpha'} \Pi_a \Pi_b, \quad \bar{T} = -\frac{\bar{L}^{ab}}{\alpha'} \bar{\Pi}_a \bar{\Pi}_b \quad (3.2a)$$

$$\bar{\partial} T = \partial \bar{T} = 0 \quad (3.2b)$$

iff the inverse inertia tensors are covariantly constant

$$\hat{\nabla}_i^+ L^{ab} = \hat{\nabla}_i^- \bar{L}^{ab} = 0 \quad (3.3)$$

where $\hat{\nabla}^\pm$ are the generalized covariant derivatives (2.18a) with torsion. Further discussion of these covariant-constancy conditions is found in Sections 5.2 and especially 5.5, which places the relations in the context of the theory of G -structures.

To study the classical Virasoro conditions, we introduce Poisson brackets in Minkowski space. The Minkowski space action of the sigma model

$$S = \frac{1}{8\pi\alpha'} \int dt d\sigma (G_{ij} + B_{ij}) \partial_+ x^i \partial_- x^j \quad (3.4a)$$

$$\partial_\pm = \partial_t \pm \partial_\sigma \quad (3.4b)$$

is obtained from the Euclidean action (2.15a) by $z = e^{\tau+i\sigma}$ and $t = i\tau$. Then general Poisson brackets may be computed from the fundamental brackets

$$[x^i(\sigma), p_j(\sigma')] = i\delta_j^i \delta(\sigma - \sigma') \quad (3.5a)$$

$$p_i = \frac{1}{4\pi\alpha'} (G_{ij} \partial_t x^j - B_{ij} \partial_\sigma x^j) \quad (3.5b)$$

where $p_i, i = 1 \dots \dim(M)$ are the canonical momenta.

We consider first the sigma-model currents

$$J_a^\pm = G_{ab} e_i^b \partial_\pm x^i \quad (3.6a)$$

$$= e_a^i (4\pi\alpha' p_i + (B_{ij} \pm G_{ij}) \partial_\sigma x^j) \quad (3.6b)$$

which are the Minkowski space analogues of the Euclidean $\Pi, \bar{\Pi}$ in (2.19e). After some algebra, we find that these currents satisfy the general current algebra

$$[J_a^+(\sigma), J_b^+(\sigma')] = 8\pi i \alpha' G_{ab} \partial_\sigma \delta(\sigma - \sigma') + 4\pi i \alpha' \delta(\sigma - \sigma') [J_c^- \hat{\omega}^{+c}_{ab} - J_c^+ \hat{\tau}^{+c}_{ab}] \quad (3.7a)$$

$$[J_a^-(\sigma), J_b^-(\sigma')] = -8\pi i \alpha' G_{ab} \partial_\sigma \delta(\sigma - \sigma') + 4\pi i \alpha' \delta(\sigma - \sigma') [J_c^+ \hat{\omega}^{-c}_{ab} - J_c^- \hat{\tau}^{-c}_{ab}] \quad (3.7b)$$

$$[J_a^+(\sigma), J_b^-(\sigma')] = 4\pi i \alpha' \delta(\sigma - \sigma') [\hat{\omega}_{ba}^{+c} J_c^+ - \hat{\omega}_{ab}^{-c} J_c^-] \quad (3.7c)$$

where $\hat{\omega}^\pm$ are the generalized connections (2.18) with torsion and the quantities

$$(\hat{\tau}^\pm)_{cab} \equiv \omega_{cab} + \omega_{abc} + \omega_{bca} \pm \frac{1}{2} H_{cab} \quad (3.8)$$

are totally antisymmetric tensors on M . We have checked that the algebra (3.7) satisfies all Jacobi identities.

We have been unable to find the general current algebra (3.7) in the literature of the sigma model, although Faddeev and Takhtajan [39] have given an example of the algebra in the special case of the principal chiral model, where

$$G_{ab} = k\eta_{ab}, \quad \omega_{ab}^c = -\frac{1}{2\sqrt{\alpha'}} f_{ab}^c, \quad H_{ab}^c = 0. \quad (3.9)$$

For the special case of the WZW model, the algebra (3.7) is closely related to the bracket form of affine $g \times g$: The currents J^+ already satisfy the algebra of affine g because $\hat{\omega}^+ = 0$ in this case, but the currents J^- need a correction factor to complete the algebra

(see eq (2.25b)). This subject is further discussed in Appendix A, which also constructs a related set of non-local chiral currents in the general sigma model.

We are now prepared to require that the chiral stress tensors

$$T_{++} = \frac{1}{8\pi\alpha'} L^{ab} J_a^+ J_b^+, \quad T_{--} = \frac{1}{8\pi\alpha'} \bar{L}^{ab} J_a^- J_b^- \quad (3.10)$$

satisfy the commuting Virasoro algebras

$$[T_{\pm\pm}(\sigma), T_{\pm\pm}(\sigma')] = \pm i [T_{\pm\pm}(\sigma) + T_{\pm\pm}(\sigma')] \partial_\sigma \delta(\sigma - \sigma') \quad (3.11a)$$

$$[T_{++}(\sigma), T_{--}(\sigma')] = 0. \quad (3.11b)$$

Using both the general current algebra (3.7) and the covariant constancy (3.3) of the inverse inertia tensors, we find after some algebra that (3.11) is satisfied iff

$$L^{ab} = 2L^{ac} G_{cd} L^{db}, \quad \bar{L}^{ab} = 2\bar{L}^{ac} G_{cd} \bar{L}^{db}. \quad (3.12)$$

These equations are the analogues on general manifolds of the high-level or classical limit (2.11) of the VME on group manifolds.

As discussed below, the classical relations (3.3) and (3.12) of this section will receive quantum corrections, e.g.

$$\hat{\nabla}_i^+ L^{ab} = \mathcal{O}(\alpha') \quad (3.13a)$$

$$L^{ab} = 2L^{ac} G_{cd} L^{db} + \mathcal{O}(\alpha') \quad (3.13b)$$

although the generalized VME in (3.13b) remains algebraic in L .

4 Summary of the unified system

We summarize here the results of our one-loop computation in the general sigma model (2.15a), postponing details of the computation until Section 6.

Including the one-loop dilatonic and counterterm contributions, the holomorphic and antiholomorphic stress tensors are

$$T = -L^{ab} \left(\frac{\Pi_a \Pi_b}{\alpha'} + \frac{1}{2} \Pi_c \Pi_d H_{ae}^c H_b^{ed} \right) + \partial(\Pi_a \Phi^a) + \mathcal{O}(\alpha') \quad (4.1a)$$

$$\bar{T} = -\bar{L}^{ab} \left(\frac{\bar{\Pi}_a \bar{\Pi}_b}{\alpha'} + \frac{1}{2} \bar{\Pi}_c \bar{\Pi}_d H_{ae}^c H_b^{ed} \right) + \bar{\partial}(\bar{\Pi}_a \bar{\Phi}^a) + \mathcal{O}(\alpha') \quad (4.1b)$$

$$a, b = 1, \dots, \dim(M) \quad (4.1c)$$

where $L^{ab} = L^{ba}$, $\bar{L}^{ab} = \bar{L}^{ba}$ are the inverse inertia tensors and $\Pi_a, \bar{\Pi}_b$ are defined in (2.19e). The second terms in T and \bar{T} are finite one-loop counterterms which characterize our renormalization scheme. The quantities Φ^a and $\bar{\Phi}^a$ in (4.1a), (4.1b) are called the

dilaton vectors, and we will see below that the dilaton vectors include the conventional dilaton as a special case.

For the holomorphic stress tensor T , we find the unified Einstein-Virasoro master equation

$$L^{cd}\hat{R}_{acdb}^+ + \hat{\nabla}_a^+\Phi_b = \mathcal{O}(\alpha') \quad (4.2a)$$

$$\Phi_a = 2L_a{}^b\Phi_b + \mathcal{O}(\alpha') \quad (4.2b)$$

$$\hat{\nabla}_i^+L^{ab} = \mathcal{O}(\alpha') \quad (4.2c)$$

$$\begin{aligned} L^{ab} = & 2L^{ac}G_{cd}L^{db} \\ & - \alpha'(L^{cd}L^{ef}H_{ce}{}^aH_{df}{}^b + L^{cd}H_{ce}{}^fH_{df}^{(a}L^{b)e)}) \end{aligned} \quad (4.2d)$$

$$- \alpha'(L^{c(a}G^{b)d}\nabla_{[c}\Phi_{d]}) + \mathcal{O}(\alpha'^2)$$

$$c = 2G_{ab}L^{ab} + 6\alpha'(2\Phi_a\Phi^a - \nabla_a\Phi^a) + \mathcal{O}(\alpha'^2) \quad (4.2e)$$

where the first line of (4.2d) is the classical master equation in (3.12).

The Virasoro conditions for the antiholomorphic stress tensor are quite similar,

$$\bar{L}^{cd}\hat{R}_{acdb}^- + \hat{\nabla}_a^-\bar{\Phi}_b = \mathcal{O}(\alpha') \quad (4.3a)$$

$$\bar{\Phi}_a = 2\bar{L}_a{}^b\bar{\Phi}_b + \mathcal{O}(\alpha') \quad (4.3b)$$

$$\hat{\nabla}_i^-\bar{L}^{ab} = \mathcal{O}(\alpha') \quad (4.3c)$$

$$\begin{aligned} \bar{L}^{ab} = & 2\bar{L}^{ac}G_{cd}\bar{L}^{db} \\ & - \alpha'(\bar{L}^{cd}\bar{L}^{ef}H_{ce}{}^aH_{df}{}^b + \bar{L}^{cd}H_{ce}{}^fH_{df}^{(a}\bar{L}^{b)e)}) \end{aligned} \quad (4.3d)$$

$$- \alpha'(\bar{L}^{c(a}G^{b)d}\nabla_{[c}\bar{\Phi}_{d]}) + \mathcal{O}(\alpha'^2)$$

$$\bar{c} = 2G_{ab}\bar{L}^{ab} + 6\alpha'(2\bar{\Phi}_a\bar{\Phi}^a - \nabla_a\bar{\Phi}^a) + \mathcal{O}(\alpha'^2) \quad (4.3e)$$

and in fact follow from the unified system (4.2) for T , using the symmetry $\Pi \leftrightarrow \bar{\Pi}$, $H \leftrightarrow -H$ of the sigma model action (2.15a).

In what follows, we refer to (4.2a) and (4.3a) as the generalized Einstein equations of the sigma model, and eqs. (4.2b) and (4.3b) are called the eigenvalue relations of the dilaton vectors. Equations (4.2d), (4.3d) are called the generalized Virasoro master equations (VMEs) of the sigma model. In Section 5.7, we show that the central charges (4.2e) and (4.3e) are consistent by Bianchi identities with the rest of the unified system. The $\mathcal{O}(\alpha')$ corrections to the covariant-constancy conditions (4.2c) and (4.3c) can be computed in principle from the solutions of the generalized VMEs.

Some simple observations

1. Algebraic form of the generalized VMEs. In parallel with the VME, the generalized VMEs (4.2d) and (4.3d) are algebraic equations for L and \bar{L} . This follows because any

derivative of L or \bar{L} can be removed (see also Section 6) by using the covariant-constancy conditions (4.2c) and (4.3c).

2. Correspondence with the VME. The non-dilatonic terms of the generalized VMEs (4.2d) and (4.3d) have exactly the form of the unimproved VME (2.4a), after the covariant substitution

$$f_{ab}{}^c \rightarrow \sqrt{\alpha'} H_{ab}{}^c \quad (4.4)$$

for the general sigma model. This correspondence is the inverse of the WZW datum in (2.22),

$$H_{ab}{}^c = \frac{1}{\sqrt{\alpha'}} f_{ab}{}^c \quad (4.5)$$

which means that, for the special case of WZW, the non-dilatonic terms of the generalized VMEs will reduce correctly to those of the unimproved VME. We return to complete the WZW reduction in Section 5.2.

3. Dilaton solution for the dilaton vector. According to the classical limit (3.12) of the generalized VMEs, one solution of the eigenvalue relations (4.2b) and (4.3b) for the dilaton vectors is

$$\Phi^a(\Phi) \equiv 2L^{ab}\nabla_b\Phi \quad (4.6a)$$

$$\bar{\Phi}^a(\Phi) \equiv 2\bar{L}^{ab}\nabla_b\Phi. \quad (4.6b)$$

In what follows, this solution is called the dilaton solution, and we shall see in the following section that the scalar field Φ is in fact the conventional dilaton of the sigma model.

5 Properties of the unified system

5.1 The conventional stress tensors of the sigma model

In this section, we check that the conventional stress tensors of the sigma model are correctly included in the unified system.

In the full system, the conventional stress tensors T_G , \bar{T}_G of the sigma model correspond to the particular solution of the generalized VMEs whose classical limit is

$$L^{ab} = \bar{L}^{ab} = L_G^{ab} = \frac{G^{ab}}{2} + \mathcal{O}(\alpha') \quad (5.1)$$

where G^{ab} is the inverse of the metric in the sigma model action. The covariant-constancy conditions (4.2c) and (4.3c) are trivially solved to this order because $\hat{\nabla}_i^\pm G_{ab} = 0$.

To obtain the form of T_G and \bar{T}_G through one loop, we must also take the dilaton solution (4.6) for the dilaton vectors, so that the dilaton contributes to the system as

$$\Phi_a = \bar{\Phi}_a = \Phi_a^G = \nabla_a\Phi + \mathcal{O}(\alpha') \quad (5.2a)$$

$$\nabla_{[a}\Phi_{b]}^G = \mathcal{O}(\alpha'). \quad (5.2b)$$

The relations (5.1) and (5.2a) then tell us that the generalized Einstein equations (4.2a) and (4.3a) simplify to the conventional Einstein equations

$$\hat{R}_{ab}^\pm - 2\hat{\nabla}_a^\pm \hat{\nabla}_b^\pm \Phi = \mathcal{O}(\alpha') \quad (5.3)$$

in agreement with equations (2.20a), (2.20b) and (2.21). Moreover, (5.2b) tells us that the dilaton terms do not contribute to the generalized VMEs in this case, and we may easily obtain

$$L^{ab} = \bar{L}^{ab} = L_G^{ab} = \frac{G^{ab}}{2} - \frac{\alpha'}{4}(H^2)^{ab} + \mathcal{O}(\alpha'^2) \quad (5.4a)$$

$$\hat{\nabla}_i^\pm L_G^{ab} = -\frac{\alpha'}{4}\hat{\nabla}_i^\pm (H^2)^{ab} + \mathcal{O}(\alpha'^2) \quad (5.4b)$$

$$T_G(\Phi) = -\frac{G^{ab}}{2\alpha'}\Pi_a\Pi_b + \partial^2\Phi + \mathcal{O}(\alpha') \quad (5.4c)$$

$$\bar{T}_G(\Phi) = -\frac{G^{ab}}{2\alpha'}\bar{\Pi}_a\bar{\Pi}_b + \bar{\partial}^2\Phi + \mathcal{O}(\alpha') \quad (5.4d)$$

by solving the generalized VMEs through the indicated order. In this case, the stress tensor counterterms in (4.1a), (4.1b) cancel against the $\mathcal{O}(\alpha')$ correction to L_G , and (5.4c), (5.4d) are consistent with (2.19). In what follows, the stress tensors $T_G(\Phi)$ and $\bar{T}_G(\Phi)$ are called the conventional stress tensors of the sigma model.

To complete the check, we evaluate the central charges $c = \bar{c} = c_G(\Phi)$ in this case,

$$c_G(\Phi) = 2G_{ab}\left(\frac{G^{ab}}{2} - \frac{\alpha'}{4}(H^2)^{ab}\right) + 6\alpha'(2|\nabla\Phi|^2 - \nabla^2\Phi) + \mathcal{O}(\alpha'^2) \quad (5.5a)$$

$$= \dim(M) + 3\alpha'(4|\nabla\Phi|^2 - 2\nabla^2\Phi - \frac{1}{6}H^2) + \mathcal{O}(\alpha'^2) \quad (5.5b)$$

$$= \dim(M) + 3\alpha'(4|\nabla\Phi|^2 - 4\nabla^2\Phi + R + \frac{1}{12}H^2) + \mathcal{O}(\alpha'^2) \quad (5.5c)$$

which agrees with the conventional central charge in (2.20c). To obtain the usual form in (5.5c), we used the conventional Einstein equations (2.20a) in the form $R = 2\nabla^2\Phi - \frac{1}{4}H^2$.

Dilaton-vector Einstein equations

For completeness, we also note the form of the system for $L = \bar{L} = L_G$ with general dilaton vectors $\Phi_a^G, \bar{\Phi}_a^G$. The dilaton vector terms still do not contribute to the generalized VMEs, whose solution is still (5.4a). For the holomorphic system, one obtains

$$T_G(\Phi_a) = -\frac{G^{ab}}{2\alpha'}\Pi_a\Pi_b + \partial(\Pi^a\Phi_a^G) + \mathcal{O}(\alpha') \quad (5.6a)$$

$$c_G(\Phi_a) = \dim(M) + 3\alpha'(4\Phi_a^G\Phi_G^a - 4\nabla_a\Phi_G^a + R + \frac{1}{12}H^2) + \mathcal{O}(\alpha'^2) \quad (5.6b)$$

$$\hat{R}_{ab}^+ - 2\hat{\nabla}_a^+\Phi_b^G = \mathcal{O}(\alpha') \quad (5.6c)$$

where Φ_G^a is unrestricted because its eigenvalue equation is trivial. The antiholomorphic system in this case is similarly obtained with $\Pi_a \rightarrow \bar{\Pi}_a$, $+\rightarrow -$ and $\Phi_a^G \rightarrow \bar{\Phi}_a^G$. In what follows, the stress tensor $T_G(\Phi_a)$ in (5.6a) will be called the conventional stress tensor with general dilaton vector, and the relations in (5.6c) will be called the dilaton-vector Einstein equations of the sigma model.

5.2 WZW and the improved VME

In this section we check that, for the special case of WZW, the unified system reduces to a holomorphic and an antiholomorphic copy of the improved VME (2.4a), where the improvement vectors D, \bar{D} are constructed from the general dilaton vectors.

We recall first from Section 4 that the non-dilatonic terms of the generalized VMEs reduce correctly to those of the VME,

$$L^{ab} = (\text{usual } L^2 \text{ and } L^2 f^2 \text{ terms}) + \text{dilaton vector term} + \mathcal{O}(\alpha'^2) \quad (5.7a)$$

$$\bar{L}^{ab} = (\text{usual } \bar{L}^2 \text{ and } \bar{L}^2 f^2 \text{ terms}) + \text{dilaton vector term} + \mathcal{O}(\alpha'^2) \quad (5.7b)$$

because $H = \frac{1}{\sqrt{\alpha'}} f$ in this case.

To study the dilaton vector terms, we recall first that the generalized Riemann tensors \hat{R}_{abcd}^\pm vanish for WZW, so the generalized Einstein equations (4.2a) and (4.3a) simplify to

$$\hat{\nabla}_a^+ \Phi_b = \mathcal{O}(\alpha'), \quad \hat{\nabla}_a^- \bar{\Phi}_b = \mathcal{O}(\alpha'). \quad (5.8)$$

These relations and the WZW datum (2.22) for H can be used to evaluate the (ordinary) covariant derivatives of the dilaton vectors

$$\nabla_a \Phi_b = \frac{1}{2\sqrt{\alpha'}} f_{ab}{}^c \Phi_c + \mathcal{O}(\alpha'), \quad \nabla_a \bar{\Phi}_b = -\frac{1}{2\sqrt{\alpha'}} f_{ab}{}^c \bar{\Phi}_c + \mathcal{O}(\alpha'), \quad (5.9)$$

so that the generalized VMEs (4.2d) and (4.3d) have the form

$$L^{ab} = (\text{usual } L^2 \text{ and } L^2 f^2 \text{ terms}) + \sqrt{\alpha'} f_{cd}{}^{(a} L^{b)c} \Phi^d + \mathcal{O}(\alpha'^2) \quad (5.10a)$$

$$\bar{L}^{ab} = (\text{usual } \bar{L}^2 \text{ and } \bar{L}^2 f^2 \text{ terms}) - \sqrt{\alpha'} f_{cd}{}^{(a} \bar{L}^{b)c} \bar{\Phi}^d + \mathcal{O}(\alpha'^2) \quad (5.10b)$$

when the sigma model is taken as WZW. Comparing with the improved VME (2.4), one is tempted to identify the constant improvement vectors D^a, \bar{D}^a as proportional by constants to the dilaton vectors $\Phi^a, \bar{\Phi}^a$. The full story is slightly more involved however, because we have not yet studied the spacetime dependence of the dilaton vectors.

Constant quantities on the group manifold

To study the coordinate dependence of Φ^a and $\bar{\Phi}^a$ for WZW, we return to solve the generalized Einstein equations (5.8) using the WZW data

$$\hat{\omega}^+{}_{ab}{}^c = 0, \quad \hat{\omega}^-{}_{ab}{}^c = -\frac{1}{\sqrt{\alpha'}} f_{ab}{}^c. \quad (5.11)$$

At leading order, the general solution of these equations can be written as conservation laws

$$\partial_i D^a = \partial_i \bar{D}^a = 0 \quad (5.12a)$$

$$D^a \equiv -\sqrt{\alpha'} \Phi^a = \text{constant} \quad (5.12b)$$

$$\bar{D}^a \equiv \sqrt{\alpha'} \bar{\Phi}^b \Omega_b{}^a = \text{constant} \quad (5.12c)$$

Here $\Omega_a{}^b$ is the adjoint action of g , whose differential equation (2.14d) was used to obtain the solution (5.12c), and we have chosen the multiplicative constants in (5.12b), (5.12c) so that the constant quantities D, \bar{D} will turn out to be the improvement vectors.

One way to understand why $\bar{\Phi}^a$ is not a constant is that \bar{L}^{ab} is not a constant either. To see this, we similarly solve the covariant-constancy conditions (4.2c), (4.3c) for the inverse inertia tensors, using the WZW data (5.11) again. At leading order the solution is

$$\partial_i L^{ab} = \partial_i \bar{L}'^{ab} = 0 \quad (5.13a)$$

$$L^{ab} = \text{constant} \quad (5.13b)$$

$$\bar{L}'^{ab} = \bar{L}^{cd} \Omega_c{}^a \Omega_d{}^b = \text{constant} \quad (5.13c)$$

where $\Omega_a{}^b$ is again the adjoint action of g .

This completes the picture for the generalized VMEs, and we see that both can be written in terms of the constant quantities[‡],

$$L^{ab} = (\text{usual } L^2 \text{ and } L^2 f^2 \text{ terms}) - f_{ce} ({}^a L^b)_c D^e + \mathcal{O}(k^{-3}) \quad (5.14a)$$

$$\bar{L}'^{ab} = (\text{usual } \bar{L}'^2 \text{ and } \bar{L}'^2 f^2 \text{ terms}) - f_{ce} ({}^a \bar{L}'^b)_c \bar{D}^e + \mathcal{O}(k^{-3}) \quad (5.14b)$$

because Ω is an inner automorphism of g and the VME is covariant under transformations in $\text{Aut}(g)$ (see Section 2.1). The result (5.14) is exactly two copies of the improved VME (2.4a), with the improvement vectors D, \bar{D} constructed from the dilaton vector in (5.12b), (5.12c).

The reason for the extra automorphism in the antiholomorphic system is in fact clear from our starting point. Both Π and $\bar{\Pi}$ are defined with the vielbein e on M , which we have identified with the left-invariant vielbein on G , but the antiholomorphic WZW current \bar{J} involves instead the right-invariant vielbein \bar{e} . This difference is responsible for the extra factor of the adjoint action Ω in the antiholomorphic part of the WZW data (2.25),

$$J_a = \frac{i}{\sqrt{\alpha'}} \Pi_a, \quad \bar{J}_a = -\frac{i}{\sqrt{\alpha'}} (\Omega^{-1})_a{}^b \bar{\Pi}_b. \quad (5.15)$$

[‡]In the high-level expansion [24, 7] of the VME, it is known that $L^{ab} = \mathcal{O}(k^{-1})$ and the $L^2 f^2$ terms of the VME are $\mathcal{O}(k^{-2})$, which corresponds to one-loop at the sigma model level. For correspondence with the loop expansion, the improvement term fLD must then also be $\mathcal{O}(k^{-2})$, which says that $D^a = \mathcal{O}(k^{-1})$. To obtain improvement vectors $D^a = \mathcal{O}(k^0)$, one must allow dilaton-vector contributions at the tree level (see also the remarks in Section 7).

Indeed, using this data and the identifications (5.12b), (5.12c) of D and \bar{D} , we find that the sigma model stress tensors (4.1a), (4.1b) reduce to the form

$$T = L^{ab} J_a J_b + i D^a \partial J_a + (L\text{-counterterm}) + \mathcal{O}(\alpha') \quad (5.16a)$$

$$\bar{T} = \bar{L}'^{ab} \bar{J}_a \bar{J}_b + i \bar{D}^a \partial \bar{J}_a + (\bar{L}'\text{-counterterm}) + \mathcal{O}(\alpha') \quad (5.16b)$$

which is nothing but two copies of the general affine-Virasoro construction.

Next, we consider the central charges (4.2e) and (4.3e), using the relations

$$\nabla_a \Phi^a = \nabla_a \bar{\Phi}^a = \mathcal{O}(\alpha') \quad (5.17)$$

which follow for WZW from eq. (5.9). Then the central charges reduce to

$$c = 2G_{ab}(L^{ab} + 6D^a D^b) + \mathcal{O}(k^{-2}) \quad (5.18a)$$

$$\bar{c} = 2G_{ab}(\bar{L}'^{ab} + 6\bar{D}^a \bar{D}^b) + \mathcal{O}(k^{-2}) \quad (5.18b)$$

in agreement with the central charge (2.4c) of the improved VME. The central charges can be taken equal $\bar{c} = c$ by choosing $\bar{L}' = L$ and $\bar{D} = D$.

We finally note that the eigenvalue relations (4.2b), (4.3b) of the dilaton vectors can be rewritten with (5.12) as

$$2L^{ab} G_{bc} D^c = D^a + \mathcal{O}(k^{-2}) \quad (5.19a)$$

$$2(\bar{L}')^{ab} G_{bc} \bar{D}^c = \bar{D}^a + \mathcal{O}(k^{-2}) \quad (5.19b)$$

which are recognized as the leading terms of two copies of the exact eigenvalue relation (2.4b) of the improved VME. This completes the one-loop check of the unified system against the improved VME.

Dilaton deformations in WZW theory

Except for their eigenvalue relations, the improvement vectors D^a , \bar{D}^a of the VME are quite general, and are in correspondence with the general dilaton vectors Φ^a , $\bar{\Phi}^a$ of the unified system with WZW background. In particular, it is well known [40] that the improvement vector D_g^a for the affine-Sugawara construction L_g is completely general,

$$D_g^a = \text{arbitrary constants} \quad (5.20)$$

because the eigenvalue relation (2.4b) is an identity in this case.

On the other hand, the dilaton solution $\Phi_a(\Phi)$ in (4.6) for the dilaton vectors provides only a restricted subset of these deformations, which in fact may be more restricted than it appears at first sight. As an example, consider the possible dilaton deformations

$$D_g^a(\Phi) = -\sqrt{\alpha'} \Phi_g^a = -\sqrt{\alpha'} \nabla_a \Phi \quad (5.21)$$

of the affine-Sugawara construction L_g . In this case we have from (5.9) that

$$\mathcal{O}(\alpha') = [\nabla_a, \nabla_b]\Phi = \frac{1}{\sqrt{\alpha'}} f_{ab}{}^c \Phi_c^g = \frac{1}{\sqrt{\alpha'}} f_{ab}{}^c \nabla_c \Phi \quad (5.22a)$$

$$\rightarrow \Phi = \text{constant}, \quad D_g^a(\Phi) = 0 \quad (5.22b)$$

so that, in contrast to (5.20), no dilaton deformations are allowed for the affine-Sugawara constructions.

Exact relations in WZW theory?

Beyond one loop, it should be emphasized that the improved VME (2.4a), (2.4b) and central charge (2.4c) are exact to all orders in this case and the quantities L^{ab} , L'^{ab} , D^a and \bar{D}^a are constant to all orders. It follows that the covariant-constancy conditions

$$\hat{\nabla}_i^+ L^{ab} = \hat{\nabla}_i^- \bar{L}^{ab} = 0 \quad (5.23a)$$

$$\hat{\nabla}_i^+ D^a = \hat{\nabla}_i^- (\bar{D}^b (\Omega^{-1})_b{}^a) = 0 \quad (5.23b)$$

are exact to all orders in the general affine-Virasoro construction. The one-loop form of (5.23a), e.g. $\hat{\nabla}_i^+ L^{ab} = \mathcal{O}(\alpha'^2)$, also follows from the one-loop master equations (5.14). This tells us that it is reasonable to assume that a higher-loop renormalization scheme can be found which preserves the VME and the relations (5.23) at arbitrary loop-order when the sigma model is taken as WZW.

If we further assume that our correspondence (5.12b), (5.12c),

$$D^a = -\sqrt{\alpha'} \Phi^a, \quad \bar{D}^a = \sqrt{\alpha'} \bar{\Phi}^b \Omega_b{}^a \quad (5.24)$$

is also exact, we are led to write down the relations

$$\hat{\nabla}_i^+ \Phi_a = \hat{\nabla}_i^- \bar{\Phi}_a = 0 \quad (5.25a)$$

$$\Phi^a (2G_{ab} L^{be} + f_{ab}{}^d L^{bc} f_{cd}{}^e) = \Phi^e \quad (5.25b)$$

$$\bar{\Phi}^a (2G_{ab} \bar{L}^{be} + f_{ab}{}^d \bar{L}^{bc} f_{cd}{}^e) = \bar{\Phi}^e \quad (5.25c)$$

as candidates for exact eigenvalue relations in WZW theory. Here, (5.25a) follows from (5.23b) and (5.24), while (5.25b), (5.25c) are transcriptions of the exact eigenvalue condition (2.4b) using (5.24). The candidate relations (5.25) take the simpler form

$$\hat{\nabla}_i^\pm \nabla_j \Phi = 0 \quad (5.26a)$$

$$\nabla^a \Phi (2G_{ab} L_g^{be} + f_{ab}{}^d L_g^{bc} f_{cd}{}^e) = \nabla^e \Phi \quad (5.26b)$$

for the dilaton of the conventional stress tensors (5.4c), where L_g^{ab} is the inverse inertia tensor (2.9a) of the affine-Sugawara construction. In this case, the literature contains some supporting evidence: The analysis of [41, 13] shows that there is a renormalization scheme in which (5.26a) holds, and, if in such a scheme the VME is also exact, then relation (5.26b) is a consequence of the exact form of L_g in (2.9).

5.3 Exact relations in the general sigma model?

Although our unified system has been computed only through one loop, we saw in Sections 5.1 and 5.2 that some of the components of the system are both

- in one-loop agreement with the conventional stress tensors of the general sigma model.
- exact for the general affine-Virasoro construction.

We collect those components here, since, with an appropriate renormalization scheme, they are candidates for exact relations in the general sigma model.

Among these candidates, we mention first the generalized VME and its central charge

$$L^{ab} = 2L^{ac}G_{cd}L^{db} - \alpha'(L^{cd}L^{ef}H_{ce}{}^aH_{df}{}^b + L^{cd}H_{ce}{}^fH_{df}{}^{(a}L^{b)e)}) \quad (5.27a)$$

$$- \alpha'(L^{c(a}G^{b)d}\nabla_{[c}\Phi_{d]})$$

$$c = 2G_{ab}L^{ab} + 6\alpha'(2\Phi_a\Phi^a - \nabla_a\Phi^a) \quad (5.27b)$$

and the corresponding antiholomorphic copy in (4.3). As discussed in the previous sections, these relations are in one-loop agreement with the conventional stress tensors of the general model, and, under the identification (5.24), they are exact to all orders in the general affine-Virasoro construction. We remark in particular that the (one-loop) generalized VMEs contain no terms proportional to the generalized Riemann tensors \hat{R}^\pm . Such terms would not affect the check against the general affine-Virasoro construction, but would invalidate (5.27a), (5.27b) as candidates for exact relations in the sigma model.

To this list, we should add the eigenvalue relations for the dilaton vectors

$$\Phi^a(2G_{ab}L^{be} + \alpha'H_{ab}{}^dL^{bc}H_{cd}{}^e) = \Phi^e \quad (5.28a)$$

$$\bar{\Phi}^a(2G_{ab}\bar{L}^{be} + \alpha'H_{ab}{}^d\bar{L}^{bc}H_{cd}{}^e) = \bar{\Phi}^e \quad (5.28b)$$

which we have constructed from the exact eigenvalue relations (2.4b) using the correspondence (5.24) and the covariant substitution $f_{ab}{}^c \rightarrow \sqrt{\alpha'}H_{ab}{}^c$ in (4.4). These relations generalize the one-loop eigenvalue relations (4.2b), (4.3b) and are in all-order agreement with the general affine-Virasoro construction.

The eigenvalue relations (5.28) take a simpler form for the conventional stress tensors $T_G(\Phi)$ of the general sigma model,

$$\nabla^a\Phi(2G_{ab}L_G^{be} + \alpha'H_{ab}{}^dL_G^{bc}H_{cd}{}^e) = \nabla^e\Phi \quad (5.29)$$

where the inverse inertia tensor L_G is given in (5.4a). Using the explicit form of L_G , we can easily check that

$$2G_{ab}L_G^{be} + \alpha'H_{ab}{}^dL_G^{bc}H_{cd}{}^e = \delta_a{}^e + \mathcal{O}(\alpha'^2) \quad (5.30)$$

so the candidate relation (5.29) is verified at two loops in the form $\nabla^e \Phi = \nabla^e \Phi + \mathcal{O}(\alpha'^2)$.

Investigation of the relations (5.27) and (5.28) at higher order is an important open problem in ICFT since such relations would, through the Bianchi identities, place strong constraints on the higher-order form of the generalized and conventional Einstein equations of the general sigma model.

5.4 Semiclassical properties of the system

The generalized VME (4.2d) can be written in the form

$$L_a{}^b = 2L_a{}^c L_c{}^b + \alpha' X(L)_a{}^b + \mathcal{O}(\alpha'^2) \quad (5.31a)$$

$$\hat{\nabla}_i^+ L_a{}^b = \mathcal{O}(\alpha') \quad (5.31b)$$

where $L_a{}^b = L^b{}_a = G_{ac} L^{cb}$ and the matrix $X_a{}^b$ is easily read from (4.2d). The solutions of this system may be studied as power series in the string tension α'

$$L_a{}^b = \frac{1}{2} P_a{}^b + \alpha' \Delta_a{}^b + \mathcal{O}(\alpha'^2) \quad (5.32a)$$

$$P_a{}^c P_c{}^b = P_a{}^b, \quad \hat{\nabla}_i^+ P_a{}^b = 0 \quad (5.32b)$$

$$\Delta_a{}^b = P_a{}^c \Delta_c{}^b + \Delta_a{}^c P_c{}^b + X(P/2)_a{}^b \quad (5.32c)$$

$$\hat{\nabla}_i^+ L_a{}^b = \alpha' \hat{\nabla}_i^+ \Delta_a{}^b + \mathcal{O}(\alpha'^2) \quad (5.32d)$$

where P is a covariantly constant projector and Δ , which must solve the linear equation (5.32c), is the $\mathcal{O}(\alpha')$ correction to L . The same forms are found for the antiholomorphic subsystem with $L \rightarrow \bar{L}$ and $P \rightarrow \bar{P}$, where \bar{P} is a covariantly constant projector which satisfies $\bar{P}^2 = \bar{P}$ and $\hat{\nabla}_i^- \bar{P} = 0$.

The zeroth-order generalized Einstein and dilaton vector equations are then

$$P^{cd} \hat{R}_{acdb}^+ + 2\hat{\nabla}_a^+ \Phi_b^{(0)} = \bar{P}^{cd} \hat{R}_{acdb}^- + 2\hat{\nabla}_a^- \bar{\Phi}_b^{(0)} = 0 \quad (5.33a)$$

$$\Phi_a^{(0)} = P_a{}^b \Phi_b^{(0)}, \quad \bar{\Phi}_a^{(0)} = \bar{P}_a{}^b \bar{\Phi}_b^{(0)}, \quad (5.33b)$$

where $\Phi_a^{(0)}$ and $\bar{\Phi}_a^{(0)}$ are the zeroth-order form of the dilaton vectors and P, \bar{P} are the covariantly constant projectors.

The classical projector P and the linear equation (5.32c) are in close correspondence with the high-level expansion [24, 7] of the VME, so we know that (5.32c) will place further restrictions on the allowed projectors and that a general solution of (5.32c) is unlikely. (See however the known solutions [7] of the VME on group manifolds and the particular solution L_G in (5.4a) on any manifold.)

On the other hand, we may easily determine $L_a{}^a = \text{Tr}(L)$ by studying $\text{Tr}(\Delta)$ and $\text{Tr}(P\Delta)$, using (5.32c). This gives the expansions

$$2L_a{}^a = \text{rank}(P) + \alpha' (P_a{}^b P_c{}^e (P_d{}^f - \frac{3}{2} \delta_d^f)) H^{acd} H_{bef} + \mathcal{O}(\alpha'^2) \quad (5.34a)$$

$$\text{rank}(P) = 4(L_a{}^b L_b{}^a - 2\alpha' L_a{}^b L_c{}^e (L_d{}^f - \delta_d^f) H^{acd} H_{bef}) + \mathcal{O}(\alpha'^2) \quad (5.34b)$$

and similarly for $L \rightarrow \bar{L}$, $P \rightarrow \bar{P}$. The relation (5.34a) generalizes the known [24] high-level expansion of the central charge in the general affine-Virasoro construction, and, using the generalized VME, the relation (5.34b) follows from (5.34a). Both relations will be useful below.

5.5 Integrability conditions and G -structures

The inverse inertia tensors L^{ab} and \bar{L}^{ab} are second-rank symmetric spacetime tensors, and we know that their associated projectors P and \bar{P} are covariantly constant

$$\hat{\nabla}_i^+ P_a{}^b = \hat{\nabla}_i^- \bar{P}_a{}^b = 0 \quad (5.35)$$

Operating with a second covariant derivative, we find that the integrability conditions

$$\hat{R}_{cd}^{+ae} P_e{}^b + \hat{R}_{cd}^{+be} P_e{}^a = 0 \quad (5.36a)$$

$$\hat{R}_{cd}^{-ae} \bar{P}_e{}^b + \hat{R}_{cd}^{-be} \bar{P}_e{}^a = 0 \quad (5.36b)$$

follow as necessary conditions for the existence of solutions to (5.35). In the following, we give two applications of the integrability conditions.

Ricci form of the generalized Einstein equations

Using (5.35) and the integrability conditions (5.36), the zeroth-order generalized Einstein equations (5.33a) are easily written in the Ricci form

$$(\hat{R}_{ab}^+ - 2\hat{\nabla}_a^+ \Phi_b^{(0)}) P^{bc} = (\hat{R}_{ab}^- - 2\hat{\nabla}_a^- \bar{\Phi}_b^{(0)}) \bar{P}^{bc} = 0 \quad (5.37)$$

where the left factors have the same form as the dilaton-vector Einstein equations (5.6c). Moreover, the steps followed to obtain the Ricci form are reversible, so that the Ricci form (5.37) and the generalized Einstein equations (5.33a) are equivalent when P and \bar{P} are covariantly constant.

Covariant derivative of $\text{Tr}(L)$

As a second application of the integrability conditions, we will verify the relations

$$\hat{\nabla}_i^+ L_a{}^a = -\frac{3}{2} \alpha' P_c{}^b H_{bed} \hat{R}_i^{+edc} + \mathcal{O}(\alpha'^2) \quad (5.38a)$$

$$\hat{\nabla}_i^- \bar{L}_a{}^a = +\frac{3}{2} \alpha' \bar{P}_c{}^b H_{bed} \hat{R}_i^{-edc} + \mathcal{O}(\alpha'^2) \quad (5.38b)$$

giving details only for (5.38a). These relations will be useful below in checking that the central charges are covariantly constant.

For the proof we note first that, according to (5.34a) and (5.35), the relation (5.38a) is equivalent to the relation

$$\hat{\nabla}_i^+(P_a{}^b P_c{}^e (P_d{}^f - \frac{3}{2}\delta_d^f) H^{acd} H_{bef}) = -3P_c{}^b H_{bed} \hat{R}_i^{+edc} \quad (5.39)$$

whose left and right sides are respectively cubic and linear in the projector P . To establish (5.39), we compute directly using $\hat{\nabla}_i^+ P = 0$, $P^2 = P$, the integrability conditions (5.36a) and the (+) form of the Bianchi identities

$$\hat{\nabla}_i^\pm H_{abc} = \pm(\hat{R}_{iab}^\pm + \hat{R}_{icab}^\pm + \hat{R}_{ibca}^\pm) \quad (5.40)$$

which follow from the explicit form of \hat{R}_{abcd}^\pm in (2.18f).

As an example, we follow that term in the differentiation (5.39) which corresponds to the first term in (5.40),

$$2P_a{}^b P_c{}^e (P_d{}^f - \frac{3}{2}\delta_d^f) H_{bef} \hat{R}_i^{+acd} \quad (5.41a)$$

$$= -2P_a{}^b (P_d{}^f - \frac{3}{2}\delta_d^f) H_{bef} \hat{R}_i^{+adc} P_c{}^e \quad (5.41b)$$

$$= 2P_a{}^b P_c{}^d (P_d{}^f - \frac{3}{2}\delta_d^f) H_{bef} \hat{R}_i^{+aec} \quad (5.41c)$$

$$= -P_a{}^b P_c{}^f H_{bef} \hat{R}_i^{+aec} \quad (5.41d)$$

where we used the integrability conditions (5.36a) to obtain (5.41c) and $P^2 = P$ to obtain (5.41d). Following similar steps for the other terms, we find that the left side of (5.39) is reduced to a quadratic form in P , and that the quadratic terms in fact cancel, verifying the right side of (5.39) and hence (5.38a). The relation (5.38b) follows similarly from the (−) form of the Bianchi identities in (5.40).

Theory of G -structures

On any manifold, there is always at least one solution to the covariant-constancy conditions (5.35) and their integrability conditions (5.36), namely

$$P^{ab} = \bar{P}^{ab} = G^{ab} \quad (5.42a)$$

$$\hat{R}_{cd}^{\pm ab} + \hat{R}_{cd}^{\pm ba} = 0 \quad (5.42b)$$

$$L^{ab} = \bar{L}^{ab} = L_G^{ab} = \frac{G^{ab}}{2} + \mathcal{O}(\alpha') \quad (5.42c)$$

where G_{ab} is the metric of the sigma model action. This solution is the classical limit of the conventional sigma model stress tensor, as discussed in Section 2.2. For WZW, the integrability conditions (5.36) are also trivially satisfied (because $\hat{R}_{abcd}^\pm = 0$) and the general solutions of the covariant-constancy conditions were obtained for this case in Section 5.2.

In general we are interested in the classification of manifolds with at least one more solution P^{ab} , beyond G^{ab} . This is a problem in the theory of G -structures [23], which

includes the study of all possible covariantly constant objects on arbitrary manifolds. Physical examples include the case of Calabi-Yau manifolds [42, 43], which are characterized by the presence of a covariantly constant spinor, and supersymmetric sigma models [44] with additional conserved currents, which are characterized by the presence of covariantly closed differential forms in target space. Here, we are looking for manifolds with one or more covariantly constant projectors P^{ab} beyond G^{ab} .

To get an idea of the properties of such manifolds, we start by examining the covariant-constancy conditions (5.35). The equation $\hat{\nabla}_i^+ P_a^b = 0$ is a first-order linear differential equation for P_a^b . This means that P_a^b is completely determined by its value at one point, say $P_a^b(x_0)$. The value of P_a^b at an arbitrary point x can then be obtained by integrating $\hat{\nabla}_i^+ P_a^b = 0$ along a curve γ from x_0 to x , or, equivalently, by parallel transport of P_a^b along γ with respect to the connection $\hat{\nabla}^+$. For consistency, the result should not depend on the choice of curve γ . Two different curves γ_1 and γ_2 give the same result if and only if we recover the original value $P_a^b(x_0)$ after parallel transport along the closed curve ζ which begins and ends at x_0 , and which is given by the union of γ_1 and $-\gamma_2$. The parallel transport of any object along a closed curve ζ can by definition be obtained from the holonomy of ζ , which is a matrix $M(\zeta) \in O(d)$. In the case of $P_a^b(x_0)$, its parallel transport along ζ is given in terms of M by

$$P_a^b(x_0) \xrightarrow{\zeta} M(\zeta)_a^c (M(\zeta)^t)_c^d P_d^b(x_0) \quad (5.43)$$

where $M(\zeta)^t$ denotes the transpose of $M(\zeta)$. Thus, the consistency requirement that (along each closed curve ζ starting and ending at x_0) $P_a^b(x_0)$ is parallel transported back to itself now reads that for all ζ

$$P_a^b(x_0) M(\zeta)_b^e = M(\zeta)_a^c P_c^e(x_0). \quad (5.44)$$

The set of all matrices $M(\zeta)$ form a group, the holonomy group [45–47, 43, 48] of the target-space manifold M , and (5.44) tells us that $\ker(P)$ is an invariant subspace of the representation $M(\zeta)_a^c$ of the holonomy group, which must therefore be reducible (except for the trivial cases when $P^{ab} = 0$ or $P^{ab} = G^{ab}$).

In the case that $\hat{\nabla}^+$ is torsion-free (i.e. $H = 0$), it is known from the de Rham global splitting theorem [49] that reducibility of the holonomy implies that the manifold must be a direct product of a certain number of submanifolds with the direct product metric, each of which has irreducible holonomy, and all possible irreducible holonomies of each submanifold have been completely classified [50–53]. As a result, the torsion-free case contains only trivial solutions of the unified system where the sigma model is a direct product of a conformal and a non-conformal field theory (see also Section 5.8). More interesting is the broader situation where the antisymmetric tensor, while still closed, is nonvanishing. In that case it is an unsolved problem which manifolds have reducible holonomy, except when the holonomy is trivial. In the latter case, the manifold must be a group manifold, and the sigma model must be a WZW model. A derivation of this fact is included in Appendix A.

To summarize, the classification of manifolds with reducible holonomy is centrally relevant for the solutions of the unified system which are solutions for generic values of α' . For any such manifold the allowed projectors P are in one-to-one correspondence with the projectors onto invariant subspaces of the representation of the holonomy group at one fixed point on the manifold. Further solutions may exist at special values of α' , but such sporadic solutions, although well-known [7] in the VME, are inaccessible in the perturbative formulation of the unified system.

5.6 Alternate forms of the central charge

Using the generalized Einstein equation and the generalized VME, the central charge (4.2e) can be written in a variety of forms,

$$c = 2L_a{}^a + 6\alpha'(2\Phi_a\Phi^a - \hat{\nabla}_a^+\Phi^a) + \mathcal{O}(\alpha'^2) \quad (5.45a)$$

$$= 4L_a{}^b L_b{}^a + 2\alpha' [L_b{}^e L_d{}^f H^{bda} H_{efa} + 3(2\Phi_a\Phi^a - \hat{\nabla}_a^+\Phi^a)] + \mathcal{O}(\alpha'^2) \quad (5.45b)$$

$$= \text{rank}(P) + 2\alpha' [L_a{}^b L_c{}^e (4L_d{}^f - 3\delta_d{}^f) H^{acd} H_{bef} + 3(2\Phi_a\Phi^a - \hat{\nabla}_a^+\Phi^a)] + \mathcal{O}(\alpha'^2) \quad (5.45c)$$

$$= \text{rank}(P) + 2\alpha' [3L^{ab}\hat{R}_{ab}^+ + L_a{}^b L_c{}^e (4L_d{}^f - 3\delta_d{}^f) H^{acd} H_{bef} + 6(\Phi_a\Phi^a - \hat{\nabla}_a^+\Phi^a)] + \mathcal{O}(\alpha'^2) \quad (5.45d)$$

$$= 4L_a{}^b L_b{}^a + 2\alpha' [3L^{ab}\hat{R}_{ab}^+ + L_a{}^b L_c{}^e H^{acd} H_{bed} + 6(\Phi_a\Phi^a - \hat{\nabla}_a^+\Phi^a)] + \mathcal{O}(\alpha'^2) \quad (5.45e)$$

and the same forms are obtained for \bar{c} by the substitution $L \rightarrow \bar{L}$, $P \rightarrow \bar{P}$, $\Phi^a \rightarrow \bar{\Phi}^a$ and $+$ $\rightarrow -$.

The form of the central charge in (5.45b) follows from (5.45a) by the generalized VME, and similarly the form in (5.45c) follows from (5.45a) by using the generalized VME in the form (5.34a). The form in (5.45d) then follows from (5.45b) by using the trace of the generalized Einstein equations (4.2a), (4.3a),

$$L^{ab}\hat{R}_{ab}^+ - \hat{\nabla}_a^+\Phi^a = \mathcal{O}(\alpha'), \quad \bar{L}^{ab}\hat{R}_{ab}^- - \hat{\nabla}_a^-\bar{\Phi}^a = \mathcal{O}(\alpha'), \quad (5.46)$$

and the final form in (5.45e) follows from (5.45d) by another application of the generalized VME in the form (5.34b).

The first form in (5.45a) is the ‘affine-Virasoro form’ of the central charge, noted above as possibly exact to all orders in the general sigma model. The form in (5.45d), with the first occurrence of the generalized Ricci tensor, will be called the ‘conventional form’ of the central charge because it reduces easily to the central charge of the conventional stress tensors

$$c_G(\Phi) = \dim(M) + 3\alpha'(4|\nabla\Phi|^2 - 4\nabla^2\Phi + R + \frac{1}{12}H^2) + \mathcal{O}(\alpha'^2) \quad (5.47)$$

when $P = G$ and $\Phi_a^G = \nabla_a\Phi$. The conventional form is also the form in which we found it most convenient to prove the constancy of c and \bar{c} (see Section 5.7). The final form of c in (5.45e) is the form which we believe comes out directly from the two-loop computation (see Section 6).

5.7 Bianchi identities and constant central charge

In this section, we check that the rest of the unified system implies that the central charges are constant

$$\partial_i c = \hat{\nabla}_i^+ c = \mathcal{O}(\alpha'^2), \quad \partial_i \bar{c} = \hat{\nabla}_i^- \bar{c} = \mathcal{O}(\alpha'^2) \quad (5.48)$$

although we give details only for c .

For this check, we use the zeroth-order form of the generalized VME and Einstein equations

$$P_a^c P_c^b = P_a^b, \quad \hat{\nabla}_i^+ P_a^b = 0 \quad (5.49a)$$

$$\beta_{ab} \equiv P^{cd} \hat{R}_{acdb}^+ + 2\hat{\nabla}_a^+ \Phi_b^{(0)} = 0 \quad (5.49b)$$

$$\Phi_a^{(0)} = P_a^b \Phi_b^{(0)} \quad (5.49c)$$

$$\hat{R}_{cd}^{+ae} P_e^b + \hat{R}_{cd}^{+be} P_e^a = 0 \quad (5.49d)$$

where (5.49d) is the integrability condition discussed in Section 5.5. We will also use the (+) form of the generalized Bianchi identities

$$\begin{aligned} & \hat{\nabla}_a^\pm \hat{R}_{bcde}^\pm + \hat{\nabla}_b^\pm \hat{R}_{cade}^\pm + \hat{\nabla}_c^\pm \hat{R}_{abde}^\pm \\ & \pm (H_{ab}^f \hat{R}_{fcde}^\pm + H_{bc}^f \hat{R}_{fade}^\pm + H_{ca}^f \hat{R}_{fbde}^\pm) = 0 \end{aligned} \quad (5.50)$$

which follow from the explicit form of the generalized Riemann tensors in (2.18f).

The form of the central charge which we will use is the ‘conventional form’ in (5.45d),

$$c = \text{rank}(P) + 6\alpha' \beta_\Phi + \mathcal{O}(\alpha'^2) \quad (5.51a)$$

$$\beta_\Phi \equiv -\frac{1}{2} P^{ab} \hat{R}_{cab}^+ + P_a^b P_c^e \left(\frac{1}{6} P_d^f - \frac{1}{4} \delta_d^f \right) H^{acd} H_{bef} + 2(\Phi_a^{(0)} \Phi_{(0)}^a - \hat{\nabla}_a^+ \Phi_{(0)}^a) \quad (5.51b)$$

so that the check is equivalent to showing that β_Φ is a constant. As we shall see, the desired relation emerges in the form

$$0 = -\hat{\nabla}_b^+ \beta_a^b + \beta_c^b H^c{}_{ba} + 2\beta_a^b \Phi_b^{(0)} = \hat{\nabla}_a^+ \beta_\Phi \quad (5.52)$$

where $\beta_{ab} = 0$ is the generalized Einstein equation.

We outline the steps needed to verify (5.52). For the first term, we find

$$\hat{\nabla}_b^+ \beta_a^b = \hat{\nabla}_b^+ (P^{cd} \hat{R}_{acd}^+ + 2\hat{\nabla}_b^+ \hat{\nabla}_a^+ \Phi_{(0)}^b) \quad (5.53a)$$

$$= \hat{\nabla}_b^+ (P^{cd} \hat{R}_{acd}^+) + 2[\hat{\nabla}_b^+, \hat{\nabla}_a^+] \Phi_{(0)}^b + 2\hat{\nabla}_a^+ \hat{\nabla}_b^+ \Phi_{(0)}^b \quad (5.53b)$$

$$\begin{aligned} &= \hat{\nabla}_a^+ \left(\frac{1}{2} P^{bc} \hat{R}_{dbc}^+ + 2\hat{\nabla}_b^+ \Phi_{(0)}^b \right) + P^{bc} (H_{af}^d \hat{R}_{dbc}^+ - \frac{1}{2} H_{cf}^d \hat{R}_{dab}^+) \\ & \quad + 2\hat{R}_{acb}^+ \Phi_{(0)}^b + 2H_{ab}^c \nabla_c \Phi_{(0)}^b \end{aligned} \quad (5.53c)$$

where we used P times the (+) Bianchi identity (5.50) and the relation (2.18e) to obtain (5.53c). For the dilaton vector term we find

$$2\beta_a{}^b\Phi_b^{(0)} = 2\hat{\nabla}_a^+(\Phi_b^{(0)}\Phi_{(0)}^b) - 2\hat{R}_{ac}^{+bd}P_d{}^c\Phi_b^{(0)} \quad (5.54a)$$

$$= 2\hat{\nabla}_a^+(\Phi_b^{(0)}\Phi_{(0)}^b) + 2\hat{R}_{ac}^{+cb}\Phi_b^{(0)} \quad (5.54b)$$

where we used (5.49c) and (5.49d) to obtain (5.54b). Adding all three terms of (5.52), we find that

$$0 = -\hat{\nabla}_b^+\beta_a{}^b + \beta^b{}_c H^c{}_{ba} + 2\beta_a{}^b\Phi_b^{(0)} \quad (5.55a)$$

$$= \hat{\nabla}_a^+(-\frac{1}{2}P^{bc}\hat{R}_{abc}^{+d} - 2\hat{\nabla}_b^+\Phi_{(0)}^b + 2\hat{\nabla}_b^+\Phi_{(0)}^b) - \frac{1}{2}\hat{R}_{ade}^{+b}P_b{}^cH_c{}^{de} \quad (5.55b)$$

$$= \hat{\nabla}_a^+\beta_\Phi \quad (5.55c)$$

where the identity (5.38a) was used to write the last term in (5.55b) as a total derivative. This completes the demonstration that the central charge c is a constant, and similar steps using the $(-)$ Bianchi identity in (5.50) show that \bar{c} is also constant.

5.8 Solution classes and a simplification

Class I and Class II solutions

The solutions of the unified system (4.2) can be divided into two classes,

Class I. T conformal but $T_G(\Phi_a)$ not conformal:

$$P^{cd}\hat{R}_{acdb}^+ + 2\hat{\nabla}_a^+\Phi_b^{(0)} = 0, \quad P_a{}^b\Phi_b^{(0)} = \Phi_a^{(0)} \quad (5.56a)$$

$$\hat{R}_{ab}^+ - 2\hat{\nabla}_a^+\Phi_b^{(0)G} \neq 0 \quad (5.56b)$$

Class II. T and $T_G(\Phi_a)$ both conformal:

$$P^{cd}\hat{R}_{acdb}^+ + 2\hat{\nabla}_a^+\Phi_b^{(0)} = 0, \quad P_a{}^b\Phi_b^{(0)} = \Phi_a^{(0)} \quad (5.57a)$$

$$\hat{R}_{ab}^+ - 2\hat{\nabla}_a^+\Phi_b^{(0)G} = 0 \quad (5.57b)$$

where $\Phi_b^{(0)G}$ is unrestricted. The distinction here is based on whether or not (in addition to the generalized Einstein equations) the dilaton-vector Einstein equations in (5.6a) are *also* satisfied. In the case when the dilaton solution $\Phi_a(\Phi)$ in (4.6) is taken for the dilaton vector, the question is whether or not the background sigma model is itself conformal in the conventional sense.

In Class I, we are constructing a conformal stress tensor T in the operator algebra of a sigma model whose conventional stress tensor $T_G(\Phi_a)$ with general dilaton vector is not conformal. This is a situation not encountered in the general affine-Virasoro construction because the conventional stress-tensor T_g of the WZW model is the affine-Sugawara construction, which is conformal. It is expected that Class I solutions are generic in the unified system, since there are so many non-conformal sigma models, but there are so far no non-trivial[§] examples (see however [54], which proposes a large set of candidates).

In Class II, we are constructing a conformal stress tensor T in the operator algebra of a sigma model whose conventional stress tensor $T_G(\Phi_a)$ with general dilaton vector is conformal. This class includes the case where the conventional stress tensors $T_G(\Phi)$ are conformal so that the sigma model is conformal in the conventional sense. The general affine-Virasoro construction provides a large set of non-trivial examples in Class II when the sigma model is the WZW action. Other examples are known from the general affine-Virasoro construction which are based on coset constructions, instead of WZW. In particular, [55] constructs exact Virasoro operators in the Hilbert space of a certain class of g/h coset constructions, and it would be interesting to study these operators as Class II solutions in the sigma model description of the coset constructions.

It is also useful to subdivide Class II solutions into Class IIa and IIb. In Class IIb, we require the natural identification

$$\Phi_a = 2L_a{}^b \Phi_b^G + \mathcal{O}(\alpha') \quad (5.58)$$

which solves (4.2b), and Class IIa is the set of solutions in Class II without this identification. Note in particular that Class IIb contains all solutions in Class II with the dilaton solution $\Phi_a(\Phi)$ in (4.6). We also remark that, for solutions in Class IIb, the Ricci form (5.37) of the generalized Einstein equations can be written as

$$(\hat{R}_{ab}^+ - 2\hat{\nabla}_a^+ \Phi_b^G)L^{bc} = \mathcal{O}(\alpha') \quad (5.59)$$

where the left factor is the dilaton-vector Einstein equation in (5.6c).

Simplification for Class IIb

For solutions in Class IIb, the unified system (4.2), (4.3) can be written in a simpler form, where the generalized Einstein equations (4.2a) and (4.3a) are replaced by the dilaton-vector Einstein equations

$$\hat{R}_{ab}^+ - 2\hat{\nabla}_a^+ \Phi_b^G = \mathcal{O}(\alpha'), \quad \hat{R}_{ab}^- - 2\hat{\nabla}_a^- \bar{\Phi}_b^G = \mathcal{O}(\alpha'). \quad (5.60)$$

This simplification of the system follows for solutions in Class IIb because the Ricci form (5.59) of the generalized Einstein equations are automatically solved when the dilaton-vector Einstein equations are satisfied (and the Ricci form is equivalent to the generalized Einstein equations when the covariant-constancy conditions (5.61c) are satisfied).

[§]Trivial examples in Class I are easily constructed as tensor products of conformal and non-conformal theories (see also Section 5.5).

A further simplification in Class IIb follows for the dilaton solution $\Phi_a(\Phi)$. In this case, equations (5.60) can be replaced by the conventional Einstein equations, so that the unified system reads

$$R_{ij} + \frac{1}{4}(H^2)_{ij} - 2\nabla_i \nabla_j \Phi = \mathcal{O}(\alpha') \quad (5.61a)$$

$$\nabla^k H_{kij} - 2\nabla^k \Phi H_{kij} = \mathcal{O}(\alpha') \quad (5.61b)$$

$$\hat{\nabla}_i^+ L^{ab} = \mathcal{O}(\alpha'), \quad \hat{\nabla}_i^- \bar{L}^{ab} = \mathcal{O}(\alpha') \quad (5.61c)$$

$$\begin{aligned} L^{ab} &= 2L^{ac} G_{cd} L^{db} \\ &- \alpha' (L^{cd} L^{ef} H_{ce}{}^a H_{df}{}^b + L^{cd} H_{ce}{}^f H_{df}{}^{(a} L^{b)e)} \\ &- \alpha' (L^{c(a} G^{b)d} \nabla_{[c} \Phi_{d]}) + \mathcal{O}(\alpha'^2) \end{aligned} \quad (5.61d)$$

$$\begin{aligned} \bar{L}^{ab} &= 2\bar{L}^{ac} G_{cd} \bar{L}^{db} \\ &- \alpha' (\bar{L}^{cd} \bar{L}^{ef} H_{ce}{}^a H_{df}{}^b + \bar{L}^{cd} H_{ce}{}^f H_{df}{}^{(a} \bar{L}^{b)e)} \\ &- \alpha' (\bar{L}^{c(a} G^{b)d} \nabla_{[c} \bar{\Phi}_{d]}) + \mathcal{O}(\alpha'^2) \end{aligned} \quad (5.61e)$$

$$c = 2G_{ab} L^{ab} + 6\alpha' (2\Phi_a \Phi^a - \nabla_a \Phi^a) + \mathcal{O}(\alpha'^2) \quad (5.61f)$$

$$\bar{c} = 2G_{ab} \bar{L}^{ab} + 6\alpha' (2\bar{\Phi}_a \bar{\Phi}^a - \nabla_a \bar{\Phi}^a) + \mathcal{O}(\alpha'^2). \quad (5.61g)$$

$$\Phi_a = \Phi_a(\Phi) = 2L_a{}^b \nabla_b \Phi, \quad \bar{\Phi}_a = \bar{\Phi}_a(\Phi) = 2\bar{L}_a{}^b \nabla_b \Phi. \quad (5.61h)$$

This form of the system is close in spirit to the VME of the general affine-Virasoro construction: The solution of the conventional Einstein equations in (5.61a), (5.61b) provides a conformal background, in which we need only look for solutions of the generalized VMEs in the form

$$L_a{}^b = \frac{P_a{}^b}{2} + \mathcal{O}(\alpha'), \quad \bar{L}_a{}^b = \frac{\bar{P}_a{}^b}{2} + \mathcal{O}(\alpha') \quad (5.62)$$

where P and \bar{P} are covariantly constant projectors. Moreover, as in the VME, we shall see below that all solutions of the simplified system (5.61) exhibit K -conjugation covariance.

5.9 K -conjugation covariance

For Class II solutions, both the generalized and the dilaton-vector Einstein equations are satisfied,

$$L^{cd} \hat{R}_{acdb}^+ + \hat{\nabla}_a^+ \Phi_b = \mathcal{O}(\alpha'), \quad \Phi_a = 2L_a{}^b \Phi_b + \mathcal{O}(\alpha') \quad (5.63a)$$

$$L_G^{cd} \hat{R}_{acdb}^+ + \hat{\nabla}_a^+ \Phi_b^G = \mathcal{O}(\alpha'), \quad (5.63b)$$

so that both the generalized and conventional stress tensors

$$T = -L^{ab} \left(\frac{\Pi_a \Pi_b}{\alpha'} + \frac{1}{2} \Pi_c \Pi_d H_{ae}{}^c H_b{}^{ed} \right) + \partial(\Pi_a \Phi^a) + \mathcal{O}(\alpha') \quad (5.64a)$$

$$T_G(\Phi_a) = -L_G^{ab}(\frac{\Pi_a \Pi_b}{\alpha'} + \frac{1}{2} \Pi_c \Pi_d H_{ae}{}^c H_b{}^{ed}) + \partial(\Pi_a \Phi_G^a) + \mathcal{O}(\alpha') \quad (5.64b)$$

are conformal. The explicit form of L_G is given in (5.4a).

Experience with the general affine-Virasoro construction leads us to suspect that Class II solutions of the unified system will exhibit K -conjugation covariance (see Section 2.1), which is the statement that the K -conjugate stress tensor \tilde{T} ,

$$\tilde{T} \equiv T_G - T, \quad \tilde{c} = c_G - c \quad (5.65)$$

is also conformal when T_G and T are conformal. Together, the conformal stress tensors T and \tilde{T} are called a K -conjugate pair, and the K -conjugate pairs $T = T_G$, $\tilde{T} = 0$ are a set of trivial examples in Class II. In what follows, we will demonstrate that (through one loop) K -conjugation covariance is indeed a property of a large subset of the solutions in Class II. A parallel demonstration can be given for the antiholomorphic sector, which is not discussed explicitly here.

We begin by translating the algebraic covariance (5.65) into the geometric system by defining the K -conjugate inverse inertia tensor \tilde{L}^{ab} ,

$$\tilde{T} = -\tilde{L}^{ab}(\frac{\Pi_a \Pi_b}{\alpha'} + \frac{1}{2} \Pi_c \Pi_d H_{ae}{}^c H_b{}^{ed}) + \partial(\Pi_a \tilde{\Phi}^a) + \mathcal{O}(\alpha'^2) \quad (5.66a)$$

$$\tilde{c} = 2G_{ab} \tilde{L}^{ab} + 6\alpha'(2\tilde{\Phi}^a \tilde{\Phi}_a - \hat{\nabla}_a^+ \tilde{\Phi}^a) + \mathcal{O}(\alpha'^2) \quad (5.66b)$$

$$\tilde{L}^{ab} = L_G^{ab} - L^{ab}, \quad \tilde{\Phi}_a = \Phi_a^G - \Phi^a \quad (5.66c)$$

where we have used (5.64) and the ‘affine-Virasoro form’ (5.45a) of the central charge to obtain (5.66b), (5.66c). The eigenvalue relation for the K -conjugate dilaton vector $\tilde{\Phi}_a$,

$$2\tilde{L}_a{}^b \tilde{\Phi}_b = \tilde{\Phi}_a + \mathcal{O}(\alpha') \quad (5.67)$$

follows from (5.63a) and (5.66c). This relation is necessary for conformal invariance of the K -conjugate theory, and, taken together with (5.57a) and (5.66c), also implies that

$$\Phi_a = 2L_a{}^b \Phi_b^G + \mathcal{O}(\alpha'), \quad \tilde{\Phi}_a = 2\tilde{L}_a{}^b \Phi_b^G + \mathcal{O}(\alpha'). \quad (5.68)$$

These relations show that K -conjugation covariance is restricted to solutions in Class IIb (see (5.58)).

To verify K -conjugation covariance, we must then show that the conjugate relations

$$\tilde{L}^{cd} \hat{R}_{acdb}^+ + \hat{\nabla}_a^+ \tilde{\Phi}_b = \mathcal{O}(\alpha') \quad (5.69a)$$

$$\hat{\nabla}_i^+ \tilde{L}^{ab} = \mathcal{O}(\alpha') \quad (5.69b)$$

$$\begin{aligned} \tilde{L}^{ab} = & 2\tilde{L}^{ac} G_{cd} \tilde{L}^{db} \\ & - \alpha'(\tilde{L}^{cd} \tilde{L}^{ef} H_{ce}{}^a H_{df}{}^b + \tilde{L}^{cd} H_{ce}{}^f H_{df}^{(a} \tilde{L}^{b)e)} \\ & - \alpha' \tilde{L}^{c(a} G^{b)d} \nabla_{[c} \tilde{\Phi}_{d]} \end{aligned} \quad (5.69c)$$

hold when the same equations are satisfied by L^{ab} and L_G^{ab} . Because the generalized Einstein equations and covariant-constancy condition are linear in L , the relations (5.69a), (5.69b) follow immediately from the corresponding relations for L and L_G .

To verify the K -conjugate VME (5.69c), we need the semiclassical expansion introduced in Section 5.4,

$$L = 2L^2 + \alpha'(X_L + X_\Phi) + \mathcal{O}(\alpha'^2) \quad (5.70a)$$

$$L = \frac{1}{2}P + \alpha'(\Delta_L + \Delta_\Phi) + \mathcal{O}(\alpha'^2) \quad (5.70b)$$

$$P^2 = P, \quad \Delta_L = \Delta_L P + P \Delta_L + X_L \quad (5.70c)$$

$$\Delta_\Phi = \Delta_\Phi P + P \Delta_\Phi + X_\Phi \quad (5.70d)$$

where $(AB)_a{}^b = A_a{}^c B_c{}^b$. Here we have divided the quantities $X = X_L + X_\Phi$ and $\Delta = \Delta_L + \Delta_\Phi$ of Section 5.4 into the ordinary (L) contributions and the dilatonic (Φ) contributions, e.g.

$$X_\Phi = \frac{1}{2}(FP - PF), \quad F_{ab} = \nabla_{[a}\Phi_{b]}^{(0)} \quad (5.71)$$

where F is the zeroth-order form of the field strength of the dilaton vector.

We also need the corresponding expansion of the K -conjugate quantities

$$\tilde{L} = \frac{1}{2}\tilde{P} + \alpha(\tilde{\Delta}_L + \tilde{\Delta}_\Phi) + \mathcal{O}(\alpha'^2) \quad (5.72a)$$

$$\tilde{P} = 1 - P, \quad \tilde{\Delta}_L = -\frac{1}{4}H^2 - \Delta_L, \quad \tilde{\Delta}_\Phi = -\Delta_\Phi \quad (5.72b)$$

$$\tilde{F} = F_G - F = \nabla_{[a}\tilde{\Phi}_{b]}^{(0)} \quad (5.72c)$$

where the relations in (5.72b), (5.72c) were obtained from (5.66c) and the explicit form of L_G in (5.4a). The absence of a G -term in $\tilde{\Delta}_\Phi$ follows because the dilaton vector fails to contribute to L_G ,

$$\Delta_\Phi^G = X_\Phi^G = 0 \quad (5.73)$$

as seen in Section 5.1.

K -conjugation covariance for the “ordinary terms” of the generalized VME is established by verifying the identity

$$\tilde{\Delta}_L \tilde{P} + \tilde{P} \tilde{\Delta}_L + \tilde{X}_L = \tilde{\Delta}_L \quad (5.74a)$$

$$\tilde{X}_L = X_G - X_L = X_L(\tilde{P}/2) \quad (5.74b)$$

using (5.72b) and the generalized VME in the form (5.70c).

To study K -conjugation for the dilatonic contribution, we evaluate the quantity

$$\tilde{\Delta}_\Phi \tilde{P} + \tilde{P} \tilde{\Delta}_\Phi + \frac{1}{2}(\tilde{F} \tilde{P} - \tilde{P} \tilde{F}) \quad (5.75a)$$

$$= \tilde{\Delta}_\Phi + \frac{1}{2}(PF_G - F_GP) \quad (5.75b)$$

using (5.72b), (5.72c) and the generalized VME in the form (5.70d). When

$$PF_G - F_GP = P^{c(a}G^{b)d}F_{cd}^G = 0 \quad (5.76)$$

holds, the relation in (5.75) is the K -conjugate form of the linear equation (5.70d) for the dilatonic contribution to L . It follows that the condition (5.76) is necessary and sufficient for K -conjugation covariance of the dilatonic contribution.

One solution of the condition (5.76) is

$$F_{ab}^G = 0 \quad (5.77)$$

which implies the dilaton solution $\Phi_a^{(0)G} = \nabla_a \Phi$ for T_G . In this case, the conditions (5.68) imply that the dilaton vectors are given by the dilaton solution $\Phi_a(\Phi)$, uniformly for T , \tilde{T} and T_G , and the unified system reduces to the simplified system (5.61).

In summary, we find that all solutions of the simplified system (5.61) exhibit K -conjugation covariance at one loop. In particular, this includes all solutions with zero dilaton vector. To find other K -conjugate pairs in Class IIb, one must find other solutions of (5.76).

We finally comment on some open questions in this direction. In the first place, it should be checked that the K -conjugate pair of stress tensors T and \tilde{T} commute at one-loop level, as expected from the general affine-Virasoro construction. Second, we remark that the discussion above does not in fact contain any explicit examples of K -conjugation covariance for the VME with non-trivial improvement: In this case, the conditions (5.76) and (5.77) read respectively

$$f_{cd}^{(a}L^{b)c}D_g^d = \mathcal{O}(k^{-3}) \quad (5.78a)$$

$$f_{ab}^c D_c^g = \mathcal{O}(k^{-2}) \quad (5.78b)$$

and, to the order at which we are working, the solution of (5.78b) is $D_g = 0$. Using the relations (5.12b) and (5.68), this then implies that $D = \tilde{D} = 0$. Therefore, the only possibility to find K -conjugation covariance in the VME with non-trivial improvement is to find solutions of (5.78a) with $D_g \neq 0$.

6 The computation

In this section, we outline the one-loop calculation which gives the unified Einstein-Virasoro master equations (4.2) and (4.3), giving details only for the holomorphic system. The corresponding results for the antiholomorphic system follow from these results by the rule noted in Section 4.

The starting point of the computation is

$$S = \frac{1}{2\alpha'} \int d^2z \Pi^a \bar{\Pi}^b (G_{ab} + B_{ab}) \quad (6.1a)$$

$$T = -\frac{1}{2\alpha'} \lambda_{ab} \Pi^a \Pi^b + \partial(\Pi^a \Phi_a) + (\text{one loop counterterms}) + \mathcal{O}(\alpha') \quad (6.1b)$$

$$\lambda_a{}^b = \lambda_a{}^c \lambda_c{}^b + \mathcal{O}(\alpha'), \quad \hat{\nabla}_c^+ \lambda_a{}^b = \mathcal{O}(\alpha') \quad (6.1c)$$

$$\langle T(z)T(w) \rangle = \frac{c/2}{(z-w)^4} + 2 \frac{\langle T(w) \rangle}{(z-w)^2} + \frac{\langle \partial T(w) \rangle}{(z-w)} + \text{reg.} \quad (6.1d)$$

where $\Pi, \bar{\Pi}$ are defined in (2.19e) and we have introduced the scaled inverse inertia tensor $\lambda_a{}^b \equiv 2L_a{}^b$. For convenience, we have also included the classical results (6.1c) of Section 3 in the starting point, though these equations can also be obtained from the tree graphs in the computation below.

By including the dilaton vector in the stress tensor instead of the action, we are following Banks, Nemeschansky and Sen [22], who showed that this language correctly reproduces the beta functions [12, 22] of the sigma model when $T \rightarrow T_G(\Phi)$ is the conventional stress tensor (5.4c) of the sigma model. Although the stress tensor method is computationally more involved than the usual action method, it is clearly the correct language in which to study the general Virasoro construction in the general sigma model.

6.1 Background field expansion

A central idea in the method of Ref. [22] is to use a covariant background field expansion to evaluate the left and right sides of (6.1d) in perturbation theory. An efficient algorithm for this expansion has been provided by Mukhi [38], generalizing earlier work in [33, 8, 9].

One way to write down this expansion is to introduce an operator M which satisfies

$$M(T_{a_1, \dots, a_n}(x)) = y^a \nabla_a T_{a_1, \dots, a_n}(x) \quad (6.2a)$$

$$M(\Pi^a) = \nabla y^a, \quad M(\bar{\Pi}^a) = \bar{\nabla} y^a \quad (6.2b)$$

$$M(\nabla y^a) = -R^a{}_{bcd}(x) y^b y^c \Pi^d, \quad M(\bar{\nabla} y^a) = -R^a{}_{bcd}(x) y^b y^c \bar{\Pi}^d, \quad (6.2c)$$

$$\nabla A^a(x) = \partial A^a(x) + \Pi^b A^c(x) \omega_{bc}{}^a, \quad \bar{\nabla} A^a(x) = \bar{\partial} A^a(x) + \bar{\Pi}^b A^c(x) \omega_{bc}{}^a \quad (6.2d)$$

$$\Pi^a = e_i{}^a \partial x^i, \quad \bar{\Pi}^a = e_i{}^a \bar{\partial} x^i \quad (6.2e)$$

where $x^i, i = 1, \dots, \dim(M)$ are the background fields and $y^a, a = 1, \dots, \dim(M)$ are the quantum fields. The background fields Π and $\bar{\Pi}$ in (6.2e) satisfy the classical equations of motion (3.1), which we use here in the form

$$\nabla \bar{\Pi}^a = \bar{\nabla} \Pi^a = -\frac{1}{2} \Pi^c \bar{\Pi}^d H_{cd}{}^a \quad (6.3)$$

where ∇ and $\bar{\nabla}$ are defined in (6.2d). Then, one uses the operator M to expand any scalar function Ψ (such as S or $\Pi_a \Phi^a$) in the form

$$\Psi = \sum_{n=0}^{\infty} \Psi^{(n)}, \quad \Psi^{(n)} = \frac{1}{n} M(\Psi^{(n-1)}) \quad (6.4)$$

where n is the number of quantum fields y .

Our goal is to evaluate the Virasoro condition (6.1d) through one loop. For this it is sufficient to compute only the fourth and second order pole terms in $\langle T(z)T(w) \rangle$ (because this and the relation $\langle T(z)T(w) \rangle = \langle T(w)T(z) \rangle$, which holds in Euclidean perturbation theory, then fix the remaining singular terms). Since the background fields $\Pi, \bar{\Pi}$ have dimension one, we need only look at the contributions to $\langle T(z)T(w) \rangle$ which involve at most two background fields. Now consider an arbitrary one-loop Feynman diagram in the expansion of $\langle T(z)T(w) \rangle$. The diagram contains a certain number $n(b, q)$ of vertices with b background fields and q quantum fields, so that

$$l = 1 + \frac{1}{2} \sum_{b, q \geq 0} n(b, q)(q - 2). \quad (6.5)$$

Ignoring the dilatonic terms in T for the moment, we see from the form of S in (6.1a) and T in (6.1b) that $b + q \geq 2$. If we denote the total number of background fields in the diagram by $b_{\text{tot}} \equiv \sum_{b, q} b n(b, q)$, we find from (6.5) that

$$b_{\text{tot}} + 2l - 2 = \sum_{b, q \geq 0} n(b, q)(b + q - 2). \quad (6.6)$$

Each of the terms on the right hand side is nonnegative, and if $n(b, q) \neq 0$ this implies that

$$b + q \leq b_{\text{tot}} + 2l. \quad (6.7)$$

In our case we have $l = 1$ and $b_{\text{tot}} \leq 2$, so that we need only those terms with $b + q \leq 4$ from the background field expansion of S and T . For the terms involving the dilaton vector, which involve an extra factor of α' , we find in a similar fashion that we need only the terms with $b + q \leq 2$. Notice that a derivative of a background field, such as $\nabla \Pi^a$, has conformal weight two and therefore counts as $b = 2$.

Using the iteration (6.4), we find for these terms the following results

$$S^{(0)} = \frac{1}{2\alpha'} \int d^2 z \Pi^a \bar{\Pi}^b (G_{ab} + B_{ab}) \quad (6.8a)$$

$$S^{(1)} = \frac{1}{2\alpha'} \int d^2 z (\Pi_a \bar{\nabla} y^a + \bar{\Pi}_a \nabla y^a + y^c H_{cab} \Pi^a \bar{\Pi}^b) \quad (6.8b)$$

$$S^{(2)} = \frac{1}{2\alpha'} \int d^2 z (\nabla y_a \bar{\nabla} y^a - \bar{\Pi}^d y^b y^c \Pi^a R_{dbca} + \frac{1}{2} y^c y^d \nabla_d H_{cab} \Pi^a \bar{\Pi}^b + \frac{1}{2} y^c H_{cab} \nabla y^a \bar{\Pi}^b + \frac{1}{2} y^c H_{cab} \Pi^a \bar{\nabla} y^b) \quad (6.8c)$$

$$S^{(3)} = \frac{1}{2\alpha'} \int d^2 z (-\frac{2}{3} \bar{\nabla} y^d y^c y^b \Pi^a R_{dcba} - \frac{2}{3} \nabla y^d y^c y^b \bar{\Pi}^a R_{dcba} + \frac{1}{3} y^c y^d \nabla_d H_{cab} \nabla y^a \bar{\Pi}^b + \frac{1}{3} y^c y^d \nabla_d H_{cab} \Pi^a \bar{\nabla} y^b + \frac{1}{3} y^c H_{cab} \nabla y^a \bar{\nabla} y^b) \quad (6.8d)$$

$$S^{(4)} = \frac{1}{2\alpha'} \int d^2 z (-\frac{1}{3} \bar{\nabla} y^d y^c y^b \nabla y^a R_{dcba} + \frac{1}{4} y^c y^d \nabla_d H_{cab} \nabla y^a \bar{\nabla} y^b) \quad (6.8e)$$

and

$$T^{(0)} = -\frac{1}{2\alpha'}\lambda_{ab}\Pi^a\Pi^b + \nabla\Pi^a\Phi_a + \Pi^a\Pi^b\nabla_b\Phi_a \quad (6.9a)$$

$$T^{(1)} = -\frac{1}{\alpha'}\lambda_{ab}\Pi^a\nabla y^b - \frac{1}{2\alpha'}\Pi^a\Pi^b y^c\nabla_c\lambda_{ab} + (\nabla\nabla y^a)\Phi_a + (\nabla y^a\Pi^b + \nabla y^b\Pi^a)\nabla_b\Phi_a \quad (6.9b)$$

$$T^{(2)} = -\frac{1}{2\alpha'}\lambda_{ab}\nabla y^a\nabla y^b - \frac{1}{\alpha'}\Pi^a\nabla y^b y^c\nabla_c\lambda_{ab} - \frac{1}{4\alpha'}\Pi^a\Pi^b y^c y^d\nabla_d\nabla_c\lambda_{ab} \\ + \frac{1}{2\alpha'}\Pi^d y^b y^c R_{dbc}{}^e \lambda_{ea}\Pi^a \quad (6.9c)$$

$$T^{(3)} = -\frac{1}{2\alpha'}\Pi^a\nabla y^b y^c y^d\nabla_d\nabla_c\lambda_{ab} - \frac{1}{2\alpha'}\nabla y^a\nabla y^b y^c\nabla_c\lambda_{ab} \\ + \frac{1}{6\alpha'}\nabla y^d y^b y^c R_{dbc}{}^e \lambda_{ea}\Pi^a + \frac{1}{2\alpha'}\Pi^d y^b y^c R_{dbc}{}^e \lambda_{ea}\nabla y^a \quad (6.9d)$$

where the superscripts in S and T indicate the number of quantum fields in the expansion. The first-order term $S^{(1)}$ of the action is zero by the classical equations of motion, and we have omitted terms with $b + q = 3$, such as $\nabla\Pi^a y^b\nabla_b\Phi_a$ and $\Pi^a y^b\Pi^c\nabla_c\nabla_b\Phi_a$ in $T^{(1)}$. We have also dropped a term proportional to $\nabla^2(y^a y^b)\nabla_a\Phi_b$ in $T^{(2)}$, and omitted $T^{(4)}$. Although such terms should in principle be included according to our previous discussion, an inspection of the table of possible Feynman diagrams in Figure 1 shows that they never appear.

The background field expansions of the action S and the stress tensor T contain many terms that depend on the spin connections. Some of them appear only in tensors like R_{abcd} and $\nabla_a H_{bcd}$ that are covariant with respect to target-space local Lorentz transformations. In Feynman diagrams, such terms yield only covariant contributions. However, there are also spin connections in ∇y^a and $\bar{\nabla} y^a$, and these give non-covariant contributions to Feynman diagrams involving the corresponding vertices. Since both S and T are Lorentz invariant, our calculation should yield a Lorentz invariant answer. The only possible non-covariant contributions come from the diagrams involving the spin connection in ∇y^a and $\bar{\nabla} y^a$, and these spin connections appear algebraically, i.e. without a derivative. There are no purely algebraic Lorentz invariant tensors one can build out of the spin-connection, and therefore, unless there are sigma model anomalies [56], all diagrams involving spin connections from ∇y^a and $\bar{\nabla} y^a$ should cancel. Sigma model anomalies are usually caused by chiral fermions, which are absent in our case, so it would seem correct to simplify the algebra by setting all spin connections in ∇y^a and $\bar{\nabla} y^a$ equal to zero. However, we have decided to check these statements by including all spin connections in our computations, and we will see that they do indeed cancel.

To proceed, it will be convenient to rearrange these expansions by eliminating ∇ in favor of ∂ using (6.2d). At the same time, we will use (6.1c) to eliminate all derivatives of λ_{ab} , and contractions between λ 's. When we do this in the expression obtained from a one-loop diagram, this induces errors only at two loops. The only case where we cannot do this is for the tree diagram (24) in Figure 1, where it would induce errors at one-loop, and we have to treat diagram (24) with a little more care.

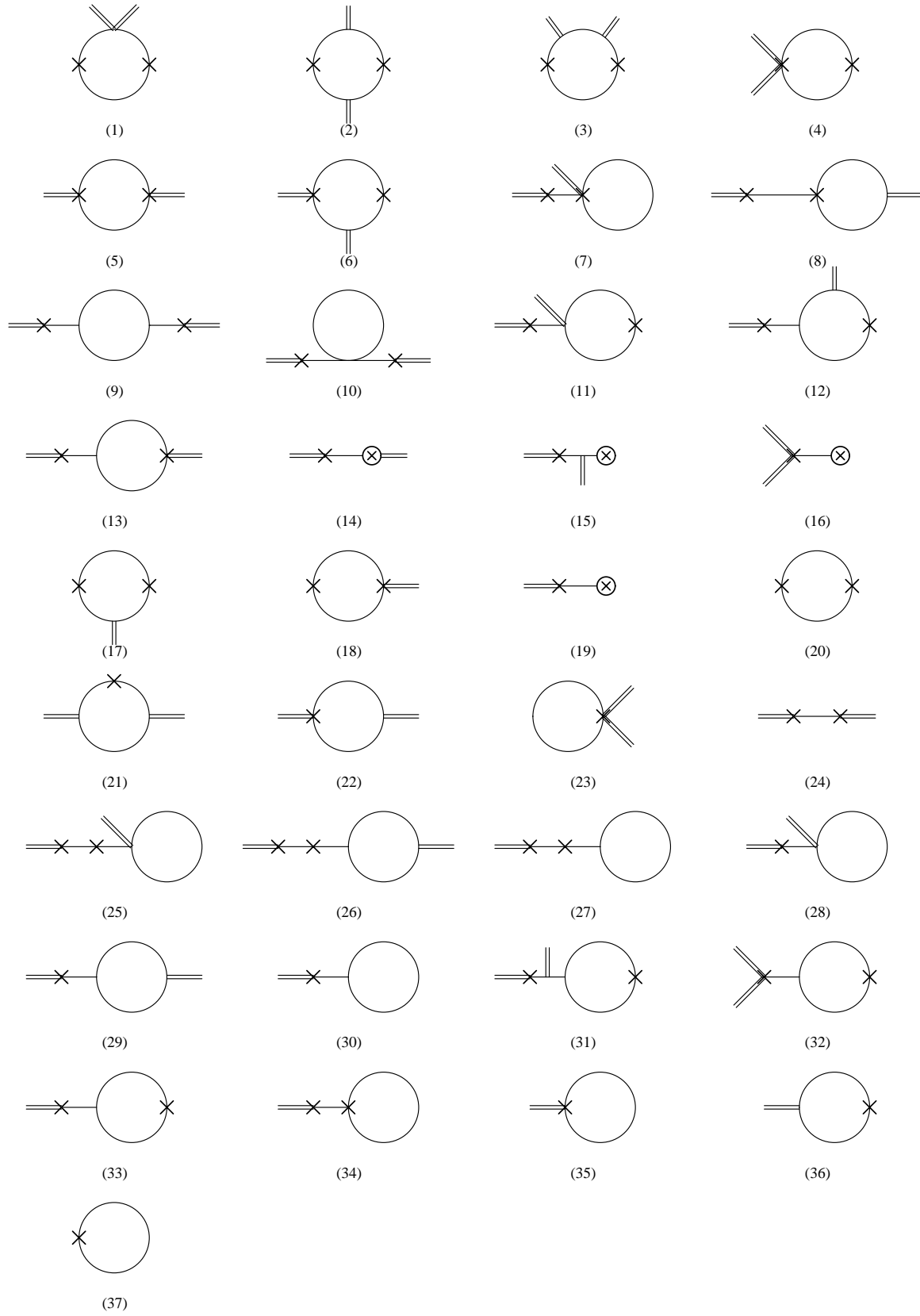


Figure 1: A table of Feynman diagrams; crosses (×) indicate vertices from the non-dilaton part of T , encircled crosses (⊗) indicate vertices coming from the dilaton part of T , double lines indicate background fields and single lines indicate quantum fields.

After this rearrangement, the expansions (6.8) and (6.9) become

$$S^{(2)} = \frac{1}{2\alpha'} \int d^2z (\partial y_a \bar{\partial} y^a - \bar{\Pi}^d y^b y^c \Pi^a R_{dbca} + \frac{1}{2} y^c y^d Z_{dcab}^1 \Pi^a \bar{\Pi}^b + y^c \hat{\omega}_{bca}^+ \partial y^a \bar{\Pi}^b - y^c \hat{\omega}_{abc}^- \Pi^a \bar{\partial} y^b) \quad (6.10a)$$

$$S^{(3)} = \frac{1}{2\alpha'} \int d^2z (-\frac{2}{3} \bar{\partial} y^d y^c y^b \Pi^a R_{dcba} - \frac{2}{3} \partial y^d y^c y^b \bar{\Pi}^a R_{dcba} + \frac{1}{3} y^c y^d Z_{dcab}^2 \partial y^a \bar{\Pi}^b + \frac{1}{3} y^c y^d Z_{dcab}^3 \Pi^a \bar{\partial} y^b + \frac{1}{3} y^c H_{cab} \partial y^a \bar{\partial} y^b) \quad (6.10b)$$

$$S^{(4)} = \frac{1}{2\alpha'} \int d^2z (-\frac{1}{3} \bar{\partial} y^d y^c y^b \partial y^a R_{dcba} + \frac{1}{4} y^c y^d \nabla_d H_{cab} \partial y^a \bar{\partial} y^b) \quad (6.10c)$$

and

$$T^{(0)} = -\frac{1}{2\alpha'} \lambda_{ab} \Pi^a \Pi^b + \partial \Pi^a \Phi_a + \Pi^a \Pi^b \partial_b \Phi_a \quad (6.11a)$$

$$T^{(1)} = -\frac{1}{\alpha'} \lambda_{ab} \Pi^a \partial y^b + \partial^2 y^a \Phi_a + \partial y^a \Pi^b (\partial_{(b} \Phi_{a)} + \omega_{[ba]}^c \Phi_c) \quad (6.11b)$$

$$T^{(2)} = -\frac{1}{2\alpha'} \lambda_{ab} \partial y^a \partial y^b + \frac{1}{2\alpha'} \Pi^a \partial y^b y^c (2\lambda_b^e \hat{\omega}_{aec}^- - \lambda_a^e H_{bec}) - \frac{1}{4\alpha'} \Pi^a \Pi^b y^c y^d Z_{dcab}^4 + \frac{1}{2\alpha'} \Pi^d y^b y^c R_{dbc}^e \lambda_{ea} \Pi^a \quad (6.11c)$$

$$T^{(3)} = -\frac{1}{2\alpha'} \Pi^a \partial y^b y^c y^d Z_{dcab}^5 - \frac{1}{\alpha'} \partial y^a \partial y^b y^c \hat{\omega}_{ca}^{+d} \lambda_{db} + \frac{1}{6\alpha'} \partial y^d y^b y^c R_{dbc}^e \lambda_{ea} \Pi^a + \frac{1}{2\alpha'} \Pi^d y^b y^c R_{dbc}^e \lambda_{ea} \partial y^a \quad (6.11d)$$

where $\hat{\omega}^\pm$ are the generalized connections (2.18b) and we have defined the following five objects

$$Z_{dcab}^1 = \nabla_d H_{cab} + 2\hat{\omega}_{ace}^- \hat{\omega}_{bd}^{+e} + \frac{1}{2} H_{ace} H_{bd}^e \quad (6.12a)$$

$$Z_{dcab}^2 = \nabla_d H_{abc} - \hat{\omega}_{ace}^- H_{db}^e - \frac{1}{2} H_{ace} H_{db}^e \quad (6.12b)$$

$$Z_{dcab}^3 = \nabla_d H_{bca} - \frac{1}{2} H_{bce} H_{da}^e + \hat{\omega}_{bce}^+ H_{da}^e \quad (6.12c)$$

$$Z_{dcab}^4 = -\lambda_{af} H_d^{ef} \hat{\omega}_{bec}^- + 2\lambda^{ef} \hat{\omega}_{afd}^- \hat{\omega}_{bec}^- + \lambda_a^e (\nabla_d H_{bec} - \hat{\omega}_{bcf}^- H_{de}^f - \frac{1}{2} H_{bcf} H_{de}^f) \quad (6.12d)$$

$$Z_{dcab}^5 = \frac{1}{4} (\lambda_a^f H_{dfe} H_{bc}^e - 2\lambda_b^f \hat{\omega}_{ace}^- H_{df}^e - 2\lambda^{ef} \hat{\omega}_{afd}^- H_{bec} - 2\lambda^{ef} H_{bfd} \hat{\omega}_{aec}^- + \frac{1}{2} \lambda_a^e \nabla_d H_{bec} + \frac{1}{2} \lambda_b^e (\nabla_d H_{aec} - \hat{\omega}_{acf}^- H_{de}^f - \frac{1}{2} H_{acf} H_{de}^f)). \quad (6.12e)$$

The background field expansions (6.10) and (6.11) are the basis for the diagrammatic computations below.

6.2 Momentum space and dimensional regularization

The term $S^{(2)}$ of the action in (6.10a) contains the kinetic term

$$S_{\text{kin}} = \frac{1}{2\alpha'} \int d^2z \partial y_a \bar{\partial} y^a \quad (6.13)$$

which defines the coordinate-space propagators

$$\langle y^a(z) y^b(w) \rangle = \alpha' G^{ab} G(z, w) \equiv -\alpha' G^{ab} \log |z - w|^2. \quad (6.14)$$

The remainder of the terms in the background field expansion (6.10) define the vertices of the theory.

It is then straightforward to write down coordinate-space expressions for the Feynman diagrams in Figure 1. In diagrams with two background fields, we can assume that the background fields are independent of the coordinates, or, in momentum space, that they have zero momentum. (In a Taylor expansion of the background fields around a fixed point on the world-sheet, all derivatives of these background fields have higher conformal weight and can be ignored.) For diagrams with one or zero background fields one has to be more careful and expand everything up to the relevant order.

In coordinate space, one then encounters various integrals not containing background fields, which one could try to evaluate using identities such as

$$\partial_{\bar{z}} \frac{1}{z - w} = \delta^{(2)}(z - w). \quad (6.15)$$

However, one soon runs into problems with coordinate-space integrals that give different answers when evaluated in different ways. This ambiguity reflects the fact that one must regularize the (relatively convergent) integrals. Although a consistent all-order coordinate-space regularization [57–59] is known for the general sigma model, and differential regularization [60] may also be applicable to this case, we have chosen instead to use the more conventional method of dimensional regularization in momentum space.

In dimensional regularization, we have to deal with several problems. First, there are terms in the action coming from the antisymmetric tensor field in (6.1a), and these contain two-dimensional antisymmetric tensors ϵ_{ij} when written in a world-sheet covariant form. Furthermore, the stress tensor (6.1b) is not part of a covariant world-sheet tensor, because \bar{T} contains $\bar{\lambda}_{ab}$ which need not be equal to λ_{ab} . We have chosen the following version of dimensional regularization to deal with these problems: We keep all interactions in exactly two dimensions, but the kinetic terms (6.13), and hence the denominators of the Feynman diagrams, will be taken in $2 - 2\epsilon$ dimensions. With such a regularization, one

has to be careful when going beyond one loop, in which case evanescent counterterms have to be taken into account (see [61] for a detailed discussion of two-loop renormalization in a two-dimensional sigma model, including problems with the ϵ tensor in two dimensions). Since our computation is limited to one loop, we will not discuss these issues further here.

In parallel with our definition $d^2z = dxdy/\pi$ we also define $d^2k = dk_x dk_y/\pi$, and with this choice we see no explicit factors of π in our calculation. Then $G(z, w)$ has the following representation

$$G(z, w) = \int d^2k \frac{e^{i\bar{k}(z-w) + ik(\bar{z}-\bar{w})}}{|k|^2} \quad (6.16)$$

and the momentum-space propagators read

$$\langle y^a(k) y^b(l) \rangle = \alpha' G^{ab} \delta^{(2)}(k + l) \frac{1}{|k|^2} \quad (6.17)$$

where k is shorthand notation for $k_z = (k_x + ik_y)/2$, the z -component of k .

To work out the Feynman diagrams we write down the corresponding expression in momentum space, continue only the denominators to $d = 2 - 2\epsilon$ dimensions, and include a factor $\Gamma(1 - \epsilon)(4\pi)^{-\epsilon}(2\pi)^{2\epsilon}\mu^{2\epsilon}$ for each loop. The μ dependent factor is the usual dimensional regularization factor, necessary to maintain naive dimensions in $d = 2$, and the μ independent factors [61] remove many unwanted constants such as the Euler constant. The d -dimensional measure $d^d k$ is the standard d -dimensional measure divided by π . The relevant integrals

$$\begin{aligned} & \Gamma(1 - \epsilon)(4\pi)^{-\epsilon}(2\pi)^{2\epsilon}\mu^{2\epsilon} \int d^d k \frac{k^a \bar{k}^b}{[|k|^2]^\alpha [|k - p|^2]^\beta} = \\ & p^{a+1-\alpha-\beta} \bar{p}^{b+1-\alpha-\beta} |p^2/\mu^2|^{-\epsilon} \frac{\Gamma[1 - \epsilon] \Gamma[1 - \epsilon - \alpha + b]}{\Gamma[\alpha] \Gamma[\beta] \Gamma[2 - \alpha - \beta + b - \epsilon]} \\ & \times \sum_{r=0}^a \binom{a}{r} \frac{\Gamma[2 - \alpha - \beta + b + r - \epsilon] \Gamma[\alpha + \beta - 1 - r + \epsilon] \Gamma[1 - \epsilon - \beta + r]}{\Gamma[2 - 2\epsilon - \alpha - \beta + r + b]} \quad (6.18) \end{aligned}$$

are easily obtained by differentiating the known result for the case $a = b = 0$.

In our one-loop calculation we encounter integrals like $\int \frac{d^2 k}{|k|^2}$, which vanish in dimensional regularization because of a cancellation between an ultraviolet and an infrared divergence. In order to check in our calculation whether infrared and ultraviolet divergences cancel separately, we have replaced the propagators in each infrared divergent diagram by

$$\frac{1}{|k|^2} \rightarrow \frac{1}{|k|^2} + \frac{1}{\zeta} \delta^{(2)}(k). \quad (6.19)$$

By taking $\zeta = \epsilon$, one can use this substitution to subtract out all infrared divergences [61], but we will keep ζ and ϵ as independent parameters. The infrared divergence of a diagram will then be given by the coefficient of $-\frac{1}{\zeta} + \frac{1}{\epsilon}$, while the ultraviolet divergence is given by the coefficient of $\frac{1}{\epsilon}$.

We have worked out all our diagrams in momentum space using (6.18) and (6.19), and then afterwards reexpressed them in coordinate space using the following translation table

$$-\frac{\bar{p}^3}{3p^3} \leftrightarrow \frac{(\bar{z} - \bar{w})^2}{(z - w)^4} \quad (6.20a)$$

$$\frac{\bar{p}^2}{2p^2} \leftrightarrow \frac{\bar{z} - \bar{w}}{(z - w)^3} \quad (6.20b)$$

$$-\frac{\bar{p}}{p} \leftrightarrow \frac{1}{(z - w)^2} \quad (6.20c)$$

$$-\frac{\bar{p}}{\zeta p} + \frac{\bar{p}}{\epsilon p} + \frac{\bar{p}}{p}(1 - \log(|p|^2/\mu^2)) \leftrightarrow \frac{G(z, w)}{(z - w)^2} \quad (6.20d)$$

$$-\frac{\bar{p}}{\zeta p} \leftrightarrow \frac{G(0, 0)}{(z - w)^2}. \quad (6.20e)$$

The first three relations follow by Fourier transformation, and the next to last relation was obtained by comparing the result for $\langle y(z)y(w) \rangle \langle \partial y(z)\partial y(w) \rangle$ in coordinate space and momentum space. On the left side of this relation, one could have in principle chosen a different finite term proportional to p/\bar{p} , but since all divergences will cancel in our calculation, this would not make any difference in the final results. Finally, the last relation was obtained from the momentum-space expression for $G(0, 0)$ as it follows from (6.16).

6.3 Diagrammatic results

We now describe the results of the calculations for each of the diagrams in Figure 1. Diagrams (1)-(13) are the one-loop contributions to the left hand side of (6.1d) involving no dilaton and two background fields, diagrams (14)-(16) are the tree-diagrams involving the dilaton with two background fields, contributing to the left hand side of (6.1d), etc. Diagrams (1)-(20) and (24) are all diagrams that contribute to $\langle T(z)T(w) \rangle$, and diagrams (21)-(23) are all diagrams that contribute to $\langle T(w) \rangle$. Diagrams (1)-(23) all have the same order of α' , and (24) has one order of α' less. For completeness, we have included in Fig. 1 a set of other diagrams (25)-(37) that one could in principle write down, but these do not contribute for a variety of reasons: The tadpole diagrams (25)-(27) cancel algebraically against (28)-(30) in (6.1d), while diagrams (31)-(33) vanish because they contain a contraction $H^{abc}\lambda_{bc}$. Finally, diagrams (34)-(37) contain integrals of the form $\int d^2p p^a \bar{p}^b$ with $a \neq b$, which vanish identically as can be seen by changing variables $p \rightarrow e^{i\phi}p$.

We now list the contributions of the individual diagrams, using the notation I_n for the n th diagram[¶] in Fig. 1. For diagrams (1)-(20) and (24), I_n is the contribution of

[¶] I_n contains the contributions from all diagrams that are topologically equivalent to the n th diagram in Fig. 1, including, when they are distinct, the mirror diagrams obtained by the interchange $z \leftrightarrow w$.

this diagram to $\langle T(z)T(w) \rangle$, while for diagrams (21)-(23) I_n is the contribution to $2 \langle T(w) \rangle / (z-w)^2$. The results come out as follows:

$$I_1 = \Pi^c \bar{\Pi}^d \frac{(\bar{z} - \bar{w})}{(z-w)^3} (\lambda^{ab} R_{bdca} - \frac{1}{2} \lambda^{ab} \nabla_a H_{bcd} - \hat{\omega}_{cea}^- \hat{\omega}_d^{+e} \lambda^{ab} - \frac{1}{4} H_{cea} H_d^e \lambda^{ab}) \quad (6.21a)$$

$$I_2 = \bar{\Pi}^c \bar{\Pi}^d \frac{(\bar{z} - \bar{w})^2}{(z-w)^4} (\frac{1}{2} \lambda^{ae} \lambda^{bf} \hat{\omega}_{cab}^+ \hat{\omega}_{def}^+) + \Pi^c \bar{\Pi}^d \frac{(\bar{z} - \bar{w})}{(z-w)^3} (-\lambda^{ae} \lambda^{bf} \hat{\omega}_{cab}^- \hat{\omega}_{def}^+) + \Pi^c \Pi^d \frac{1}{(z-w)^2} (\frac{1}{2} \lambda^{ae} \lambda^{bf} \hat{\omega}_{cab}^- \hat{\omega}_{def}^-) \quad (6.21b)$$

$$I_3 = \bar{\Pi}^c \bar{\Pi}^d \frac{(\bar{z} - \bar{w})^2}{(z-w)^4} (-\frac{1}{2} \lambda^{ab} \hat{\omega}_{caf}^+ \hat{\omega}_{db}^{+f}) + \Pi^c \bar{\Pi}^d \frac{(\bar{z} - \bar{w})}{(z-w)^3} (2 \lambda^{ab} \hat{\omega}_{caf}^- \hat{\omega}_{db}^{+f}) + \Pi^c \Pi^d \frac{G(z, w) - \frac{3}{2}}{(z-w)^2} (-\lambda^{ab} \hat{\omega}_{caf}^- \hat{\omega}_{db}^{-f}) \quad (6.21c)$$

$$I_4 = \Pi^c \Pi^d \frac{1}{(z-w)^2} (\frac{1}{2} \lambda^{ab} \lambda_c^f H_{feb} \hat{\omega}_d^{-e}{}_a + \lambda^{ab} \lambda^{ef} \hat{\omega}_{cfb}^- \hat{\omega}_{dea}^- + \frac{1}{2} \lambda_c^e \lambda^{ab} \nabla_b H_{dea} + \frac{1}{2} \lambda_c^e \lambda^{ab} \hat{\omega}_{dfa}^- H_e^f{}_b + \frac{1}{4} \lambda_c^e \lambda^{ab} H_{dfa} H_e^f{}_b - \lambda^{ab} \lambda_c^e R_{bdea}) \quad (6.21d)$$

$$I_5 = \Pi^c \Pi^d \frac{1}{(z-w)^2} (\frac{1}{4} \lambda_c^e \lambda_d^f H_{eab} H_f^{ab} + \lambda_c^e \lambda^{af} H_{eab} \hat{\omega}_{df}^{-b} + \lambda^{be} \lambda^{af} \hat{\omega}_{cea}^- \hat{\omega}_{dbf}^-) + \Pi^c \Pi^d \frac{G(z, w)}{(z-w)^2} (-\frac{1}{4} \lambda_c^e \lambda_d^f H_{eab} H_f^{ab} - \lambda_c^e \lambda^{bf} H_{eba} \hat{\omega}_{df}^{-a} - \lambda^{ef} \hat{\omega}_{cea}^- \hat{\omega}_{df}^{-a}) \quad (6.21e)$$

$$I_6 = \Pi^c \bar{\Pi}^d \frac{(\bar{z} - \bar{w})}{(z-w)^3} (2 \lambda^{ab} \lambda^{ef} \hat{\omega}_{ceb}^- \hat{\omega}_{dfa}^+ - 2 \lambda^{be} \hat{\omega}_{cef}^- \hat{\omega}_{db}^{+f}) + \Pi^c \Pi^d \frac{1}{(z-w)^2} (-\lambda^{ab} \lambda_c^e H_{ebf} \hat{\omega}_{da}^{-f} - 2 \lambda^{ab} \lambda^{ef} \hat{\omega}_{ceb}^- \hat{\omega}_{dfa}^-) + \Pi^c \Pi^d \frac{G(z, w) - 1}{(z-w)^2} (2 \lambda^{be} \hat{\omega}_{cef}^- \hat{\omega}_{db}^{-f} + \lambda^{ab} \lambda_c^e H_{ebf} \hat{\omega}_{da}^{-f}) \quad (6.21f)$$

$$I_7 = \Pi^c \Pi^d \frac{G(0, 0)}{(z-w)^2} (\frac{1}{4} \lambda_d^b \lambda_c^f H_{bea} H_f^e{}_a - \frac{1}{2} \lambda_d^f H_{fea} \hat{\omega}_c^{-ea} + \lambda_d^b \lambda^{ef} \hat{\omega}_{cfa}^- H_{be}^a - \frac{1}{2} \lambda_d^e \nabla_a H_{ce}^a - \frac{1}{2} \lambda_d^e \hat{\omega}_{cfa}^- H_e^{fa} - \frac{1}{4} \lambda_d^e H_{cfa} H_e^{fa})$$

$$+\frac{1}{3}\lambda_d^b\lambda_c^e R_{abe}{}^a + \lambda_d^e R_{ace}{}^a) \quad (6.21g)$$

$$I_8 = \Pi^c \Pi^d \frac{G(0,0)}{(z-w)^2} (-\hat{\omega}_{cea}^- H_b{}^e{}_f \lambda^{af} \lambda_d^b + \lambda_d^f \hat{\omega}_{cea}^- H_f{}^{ea}) \quad (6.21h)$$

$$I_9 = \Pi^c \Pi^d \frac{G(z,w)-1}{(z-w)^2} (-\frac{1}{4}\lambda_c^a \lambda_d^b H_{aef} H_b{}^{ef}) \\ + \Pi^c \Pi^d \frac{G(0,0)}{(z-w)^2} (\frac{1}{4}\lambda_c^a \lambda_d^b H_{aef} H_b{}^{ef}) \quad (6.21i)$$

$$I_{10} = \Pi^c \Pi^d \frac{G(0,0)}{(z-w)^2} (-\frac{1}{3}\lambda_c^a \lambda_d^e R_{bea}{}^b) \quad (6.21j)$$

$$I_{11} = \Pi^c \Pi^d \frac{1}{(z-w)^2} (\lambda_c^f \lambda^{be} R_{efdb} - \frac{1}{2}\lambda_c^f \lambda^{be} \nabla_e H_{dfb} \\ - \frac{1}{2}\lambda_c^f \lambda^{be} \hat{\omega}_{dab}^- H_f{}^a{}_e - \frac{1}{4}\lambda_c^f \lambda^{be} H_{dab} H_f{}^a{}_e) \\ + \Pi^c \bar{\Pi}^d \frac{(\bar{z}-\bar{w})}{(z-w)^3} (-2\lambda_c^f \lambda^{be} R_{efdb} + \lambda_c^f \lambda^{be} \nabla_e H_{dbf} \\ + \frac{1}{2}\lambda_c^f \lambda^{be} H_{dab} H_f{}^a{}_e - \lambda_c^f \lambda^{be} \hat{\omega}_{dab}^+ H_f{}^a{}_e) \quad (6.21k)$$

$$I_{12} = \Pi^c \bar{\Pi}^d \frac{(\bar{z}-\bar{w})}{(z-w)^3} (\lambda_c^b \lambda^{ah} H_{bfa} \hat{\omega}_d{}^{+f}{}_h) \\ + \Pi^c \Pi^d \frac{G(z,w)-\frac{3}{2}}{(z-w)^2} (-\lambda_c^b \lambda^{ah} H_{baf} \hat{\omega}_{dh}^-{}^f) \\ + \Pi^c \Pi^d \frac{G(0,0)}{(z-w)^2} (\lambda_c^b \lambda^{ah} H_{baf} \hat{\omega}_{dh}^-{}^f) \quad (6.21l)$$

$$I_{13} = \Pi^c \Pi^d \frac{G(z,w)-1}{(z-w)^2} (\frac{1}{2}\lambda_c^b \lambda_d^f H_{bae} H_f{}^{ae} + \lambda_c^b \lambda^{af} H_{bae} \hat{\omega}_{df}^-{}^e) \\ + \Pi^c \Pi^d \frac{G(0,0)}{(z-w)^2} (-\frac{1}{2}\lambda_c^b \lambda_d^f H_{bae} H_f{}^{ae} - \lambda_c^b \lambda^{af} H_{bae} \hat{\omega}_{df}^-{}^e) \quad (6.21m)$$

$$I_{14} = \frac{1}{(z-w)^2} (4\Pi^c \lambda_c^a \partial(\Phi_a) + 2\Pi^c \Pi^d \lambda_c^a \nabla_{[a} \Phi_{d]}) \quad (6.21n)$$

$$I_{15} = \Pi^c \bar{\Pi}^d \frac{(\bar{z}-\bar{w})}{(z-w)^3} (\lambda_c^a \Phi_b \hat{\omega}_d{}^{+ab}) \\ + \Pi^c \Pi^d \frac{1}{(z-w)^2} (-2\lambda_c^a \Phi_b \hat{\omega}_d{}^{-ab}) \quad (6.21o)$$

$$I_{16} = +\Pi^c \Pi^d \frac{1}{(z-w)^2} (\lambda_c^a \Phi_b \hat{\omega}_d{}^{-ab} + \lambda_d^a \Phi_b \hat{\omega}_c{}^{-ab}) \quad (6.21p)$$

$$I_{17} = \bar{\Pi}^c \bar{\Pi}^d \frac{(\bar{z}-\bar{w})^2}{(z-w)^4} (\lambda^{fb} \hat{\omega}_{cef}^+ \hat{\omega}_d{}^{+e}{}_b - \lambda^{eb} \lambda^{af} \hat{\omega}_{dab}^+ \hat{\omega}_{cfe}^+)$$

$$\begin{aligned}
& +\Pi^c\bar{\Pi}^d\frac{(\bar{z}-\bar{w})}{(z-w)^3}(-\lambda^{fb}\hat{\omega}_{cef}^-\hat{\omega}_d^{+e}{}_b+\lambda^{eb}\lambda^{af}\hat{\omega}_{dab}^+\hat{\omega}_{cfe}^- \\
& \quad +\lambda^{fb}\hat{\omega}_{cef}^+\hat{\omega}_d^{+e}{}_b-\lambda^{eb}\lambda^{af}\hat{\omega}_{dab}^+\hat{\omega}_{cfe}^+) \\
& +\Pi^c\Pi^d\frac{1}{(z-w)^2}(-\lambda^{fb}\hat{\omega}_{cef}^-\hat{\omega}_d^{+e}{}_b+\lambda^{eb}\lambda^{af}\hat{\omega}_{dab}^+\hat{\omega}_{cfe}^-)
\end{aligned} \tag{6.21q}$$

$$\begin{aligned}
I_{18} &= \Pi^c\bar{\Pi}^d\frac{(\bar{z}-\bar{w})}{(z-w)^3}(2\lambda^{fb}\hat{\omega}_{cef}^-\hat{\omega}_d^{+e}{}_b-2\lambda^{eb}\lambda^{af}\hat{\omega}_{dab}^+\hat{\omega}_{cfe}^-) \\
& +\Pi^c\Pi^d\frac{1}{(z-w)^2}(2\lambda^{fb}\hat{\omega}_{cef}^-\hat{\omega}_d^{+e}{}_b-2\lambda^{eb}\lambda^{af}\hat{\omega}_{dab}^+\hat{\omega}_{cfe}^-)
\end{aligned} \tag{6.21r}$$

$$\begin{aligned}
I_{19} &= \Pi^c\Pi^d\frac{1}{(z-w)^2}(2\lambda_c{}^e\hat{\omega}_d^{+b}{}_e\Phi_b+2\lambda^{be}\hat{\omega}_{dce}^+\Phi_b) \\
& +\frac{1}{(z-w)^2}(+2\lambda_c{}^b\partial\Pi^c\Phi_b-2\lambda_c{}^b\Pi^c\partial\Phi_b) \\
& +\Pi^c\bar{\Pi}^d\frac{(\bar{z}-\bar{w})}{(z-w)^3}(2\lambda_c{}^e\hat{\omega}_d^{+b}{}_e\Phi_b) \\
& +\frac{(\bar{z}-\bar{w})}{(z-w)^3}(-2\lambda_c{}^b\Pi^c\bar{\partial}\Phi_b)
\end{aligned} \tag{6.21s}$$

$$\begin{aligned}
I_{20} &= \frac{1}{(z-w)^4}(\frac{1}{2}\lambda_a{}^b\lambda_b{}^a) \\
& +\bar{\Pi}^c\bar{\Pi}^d\frac{(\bar{z}-\bar{w})^2}{(z-w)^4}(-\frac{1}{2}\lambda^{fb}\hat{\omega}_{cef}^+\hat{\omega}_d^{+e}{}_b+\frac{1}{2}\lambda^{eb}\lambda^{af}\hat{\omega}_{dab}^+\hat{\omega}_{cfe}^+) \\
& +\Pi^c\bar{\Pi}^d\frac{(\bar{z}-\bar{w})}{(z-w)^3}(-\lambda^{fb}\hat{\omega}_{cef}^+\hat{\omega}_d^{+e}{}_b+\lambda^{eb}\lambda^{af}\hat{\omega}_{dab}^+\hat{\omega}_{cfe}^+) \\
& +\Pi^c\Pi^d\frac{1}{(z-w)^2}(-\frac{1}{2}\lambda^{fb}\hat{\omega}_{cef}^+\hat{\omega}_d^{+e}{}_b+\frac{1}{2}\lambda^{eb}\lambda^{af}\hat{\omega}_{dab}^+\hat{\omega}_{cfe}^+)
\end{aligned} \tag{6.21t}$$

$$I_{21} = \Pi^c\Pi^d\frac{G(0,0)}{(z-w)^2}(-\lambda^{ab}\hat{\omega}_{cae}^-\hat{\omega}_{db}^-{}_e) \tag{6.21u}$$

$$I_{22} = \Pi^c\Pi^d\frac{G(0,0)}{(z-w)^2}(\lambda_c{}^e\hat{\omega}_{dab}^-H_e{}^{ab}+2\lambda^{be}\hat{\omega}_{dab}^-\hat{\omega}_c{}^{-a}{}_e) \tag{6.21v}$$

$$\begin{aligned}
I_{23} &= \Pi^c\Pi^d\frac{G(0,0)}{(z-w)^2}(\lambda_c{}^eR_{bde}{}^b-\frac{1}{2}\lambda_c{}^fH_{fea}\hat{\omega}_d^{-ea}-\lambda^{ef}\hat{\omega}_{cfa}^-\hat{\omega}_{de}^-{}_a \\
& \quad -\frac{1}{2}\lambda_c{}^e\nabla_aH_{de}{}^a-\frac{1}{2}\lambda_c{}^e\hat{\omega}_{dfa}^-H_e{}^{fa}-\frac{1}{4}\lambda_c{}^eH_{dfa}H_e{}^{fa})
\end{aligned} \tag{6.21w}$$

$$I_{24} = \Pi^c\Pi^d\frac{1}{(z-w)^2}(-\frac{1}{\alpha'}\lambda_c{}^a\lambda_{da}). \tag{6.21x}$$

The background fields in these results are evaluated at the point (w, \bar{w}) .

6.4 Derivation of the unified system

With the diagrammatic results (6.21), we can now obtain the unified Einstein-Virasoro master equation at the one-loop level. If we insert the results from the diagrams into (6.1d), and we include also the classical expectation value of T

$$\langle T(w) \rangle_{\text{cl}} = T^{(0)}(w) \equiv -\frac{1}{2\alpha'} \lambda_{ab} \Pi^a \Pi^b + \partial(\Pi^a \Phi_a), \quad (6.22)$$

we obtain an equation of the form

$$\mathcal{O}(\alpha') = \langle T(z)T(w) \rangle - \frac{c/2}{(z-w)^4} - 2 \frac{\langle T(w) \rangle}{(z-w)^2} \quad (6.23a)$$

$$= A \frac{1}{(z-w)^4} + (\Pi^c \bar{\Pi}^d B_{cd}) \frac{(\bar{z} - \bar{w})}{(z-w)^3} + (\Pi^c \Pi^d C_{cd} + \partial(\Pi^c E_c)) \frac{1}{(z-w)^2} \quad (6.23b)$$

in which, quite remarkably, all terms proportional to $(\bar{z} - \bar{w})^2/(z-w)^4$, $G(z, w)$ and $G(0, 0)$ cancel in a highly non-trivial fashion. The fact that the infrared and ultraviolet divergences cancel separately, which one would naively expect for a conserved current like T , is an excellent check on the diagrammatic results (6.21). After a certain amount of algebra, we find the following expressions for A , B_{cd} , C_{cd} and E_c

$$A = \frac{1}{2} \lambda_a^b \lambda_b^a - \frac{c}{2} = \mathcal{O}(\alpha') \quad (6.24a)$$

$$B_{cd} = \lambda^{ab} (\hat{R}_{dabc}^+ - 2\lambda_c^f \hat{R}_{dabf}^+) - 2\lambda_c^b \hat{\nabla}_d^+ \Phi_b = \mathcal{O}(\alpha') \quad (6.24b)$$

$$C_{cd} = \frac{1}{\alpha'} (\lambda_{cd} - \lambda_c^a \lambda_{ad}) + \frac{1}{2} \lambda^{ae} \lambda^{bf} H_{cab} H_{def} - \frac{1}{2} \lambda^{ef} H_{cbe} H_d^b{}_f + \lambda_c^a \nabla_{[a} \Phi_{d]} + \lambda_d^a \nabla_{[a} \Phi_{c]} = \mathcal{O}(\alpha') \quad (6.24c)$$

$$E_c = 2(\lambda_c^a \Phi_a - \Phi_c) = \mathcal{O}(\alpha') \quad (6.24d)$$

Each of these tensors must vanish separately at the indicated order because they multiply independent structures in (6.23b). These covariant results show that all spin connections, and hence anomalies, cancel in the final result, as expected in the absence of chiral fermions.

Eigenvalue relation for the dilaton vector

Eq. (6.24d) is recognized (with $\lambda_a^b = 2L_a^b$) as the eigenvalue relation (4.2b) of the dilaton eigenvector. Using (6.1c), a particular solution of the eigenvalue relation is the dilaton solution $\Phi_a = \Phi_a(\Phi) = 2L_a^b \nabla_b \Phi$ in (4.6).

Generalized Einstein equations

From (6.24b) we derive that $B_{cd} - 2\lambda_c^g B_{gd} = \mathcal{O}(\alpha')$, which yields

$$\lambda^{ab} \hat{R}_{dabc}^+ + 2\lambda_c^b \hat{\nabla}_d^+ \Phi_b = \mathcal{O}(\alpha') \quad (6.25)$$

as an equivalent form of (6.24b). Using (6.24d) and (6.1c), we find that (6.25) is also equivalent to

$$\lambda^{ab} \hat{R}_{dabc}^+ + 2 \hat{\nabla}_d^+ \Phi_c = \mathcal{O}(\alpha') \quad (6.26)$$

which is recognized as the generalized Einstein equation (4.2a).

Central charge

In (6.24a) we recognize the classical limit of the central charge in (4.2e), which was in fact determined by using the Bianchi identities and the fact that the central charges must be constant (see Section 5.7). It should also be possible to verify the order α' terms of (4.2e) in a two-loop calculation, but we have not attempted to do this because the renormalization of the sigma model becomes quite subtle at two loops.

We can however make a few simple remarks in this direction. In addition to the two-loop contributions, there are also one-loop diagrams with one dilaton vector insertion and tree diagrams with two dilaton vector insertions. These diagrams are easily evaluated and we find that they contribute to the coefficient A in the form

$$A = \frac{1}{2} \lambda_a^b \lambda_b^a - \frac{c}{2} \quad (6.27a)$$

$$+ 6\alpha' (\Phi_a \Phi^a - \hat{\nabla}_a^+ \Phi^a) \quad (6.27b)$$

$$+ \text{two-loop contributions} \quad (6.27c)$$

$$= \mathcal{O}(\alpha'^2). \quad (6.27d)$$

Looking at the alternate forms of the central charge in (5.45), we see only one form consistent with these terms in A , namely (5.45e). Thus, we expect that a complete two-loop computation of the central charge will give the form (5.45e) directly.

Generalized VME

This leaves equation (6.24c), which, except in the classical limit, does not agree in form with the generalized VME in (4.2d), and, in particular, (6.24c) does not agree with the VME when the sigma model is the WZW action. The VME is derived using OPEs, which is a particular way of regularizing the WZW model. Since different regularization schemes can differ only by finite local counterterms, we are led to look for a finite local counterterm which restores the form of the VME. Such a counterterm is easily found, namely if we define

$$T' = T - \frac{1}{4} \Pi^c \Pi^d \lambda^{ef} H_{dbf} H_c^b{}^e \quad (6.28)$$

and use T' rather than T in (6.1d), we find a few extra contributions to C_{cd} . In particular, we have to include the counterterm in the $T^{(0)}(w)$ contribution to $\langle T(w) \rangle$, and in addition there are tree diagrams with an insertion of the counterterm contributing to

$\langle T(z)T(w) \rangle$. With these extra terms, the coefficient C_{cd} becomes

$$\begin{aligned}
C_{cd} &= \frac{1}{\alpha'}(\lambda_{cd} - \lambda_c^a \lambda_{ad}) + \frac{1}{2}\lambda^{ae}\lambda^{bf}H_{cab}H_{def} \\
&\quad - \frac{1}{2}\lambda^a_{(c}H_{d)be}H_a^b{}_f\lambda^{ef} + \lambda_c^a\nabla_{[a}\Phi_{d]} + \lambda_d^a\nabla_{[a}\Phi_{c]} \\
&= \mathcal{O}(\alpha')
\end{aligned} \tag{6.29}$$

which is recognized as the generalized VME (4.2d). This concludes the derivation of the unified Einstein-Virasoro master equation at the one-loop level.

7 Conclusions

We have studied the general Virasoro construction

$$T = -\frac{L_{ij}}{\alpha'}\partial x^j\partial x^i + \mathcal{O}(\alpha'^0) \tag{7.1a}$$

$$i, j = 1, \dots, \dim(M) \tag{7.1b}$$

at one loop in the operator algebra of the general non-linear sigma model, where L is a spin-two spacetime tensor field called the inverse inertia tensor. The construction is summarized by a unified Einstein-Virasoro master equation which describes the covariant coupling of L to the spacetime fields G , B and Φ_a , where G and B are the metric and antisymmetric tensor of the sigma model and Φ_a is the dilaton vector, which generalizes the derivative $\nabla_a\Phi$ of the dilaton Φ . As special cases, the unified system contains the Virasoro master equation of the general affine-Virasoro construction and the conventional Einstein equations of the canonical sigma model stress tensors. More generally, the unified system describes a space of conformal field theories which is presumably much larger than the sum of these two special cases.

We have also discussed a number of algebraic and geometric properties of the system, noting in particular the relation of the system to an unsolved problem in the theory of G -structures on manifolds with torsion.

In addition to questions posed in the text, we list here a number of other important directions.

1. New solutions. It is important to find new solutions of the unified system, beyond the canonical stress tensors of the sigma model and the general affine-Virasoro construction. Here the question of G -structures (see Section 5.5) and solution classes (see Section 5.8) will be important.
2. Duality. The unified system contains the coset constructions in two distinct ways, that is, both as $G_{ab} = k\eta_{ab}$, $L^{ab} = L_{g/h}^{ab}$ in the general affine-Virasoro construction and among the canonical stress tensors of the sigma model with the sigma model metric that corresponds to the coset construction. This is an indicator of new duality transformations in

the system, possibly exchanging L and G , which may go beyond the coset constructions. In this connection, we remind the reader that the VME has been identified [2] as an Einstein-Maxwell system with torsion on the group manifold, where the inverse inertia tensor is the inverse metric on tangent space. Following this hint, it may be possible to cast the unified system on group manifolds as two coupled Einstein systems, with exact covariant constancy of both G and L .

3. Spacetime action and/or C -function. These have not yet been found for the unified system, but we remark that they are known for the special cases unified here: The spacetime action [12, 62] is known for the conventional Einstein equations of the sigma model, and, for this case, the C -function is known [13] for constant dilaton. Moreover, an exact C -function is known [63] for the special case of the unimproved VME.

4. World-sheet actions. We have studied here only the Virasoro operators constructible in the operator algebra of the general sigma model, but we have not yet worked out the world-sheet actions of the corresponding new conformal field theories, whose beta functions should be the unified Einstein-Virasoro master equation. This is a familiar situation in the general affine-Virasoro construction, whose Virasoro operators are constructed in the operator algebra of the WZW model, while the world-sheet actions of the corresponding new conformal field theories include spin-one [28] gauged WZW models for the coset constructions and spin-two [26, 29, 30] gauged WZW models for the generic construction.

As a consequence of this development in the general affine-Virasoro construction, more or less standard Hamiltonian methods now exist for the systematic construction of the new world-sheet actions from the new stress tensors, and we know for example that K -conjugation covariance is the source of the spin-two gauge invariance in the generic case. At least at one loop, a large subset of Class IIb solutions of the unified system exhibit K -conjugation covariance (see Section 5.9), so we may reasonably expect that the world-sheet actions for generic constructions in this subset are spin-two gauged sigma models. For solutions with no K -conjugation covariance, the possibility remains open that these constructions are dual descriptions of other conformal sigma models.

5. Superconformal extensions. The method of Ref. [22] has been extended [64–66] to the canonical stress tensors of the supersymmetric sigma model. The path is therefore open to study general superconformal constructions in the operator algebra of the general sigma model with fermions. Such superconformal extensions should then include and generalize the known $N = 1$ and $N = 2$ superconformal master equations [67] of the general affine-Virasoro construction.

In this connection, we should mention that that the Virasoro master equation is the true master equation, because it includes as a small subspace all the solutions of the superconformal master equations. It is reasonable to expect therefore that, in the same way, the unified system of this paper will include the superconformal extensions.

6. Tree-level dilaton vectors. We have assumed in our computation that the dilaton vector, like the dilaton, satisfies $\Phi_a = \mathcal{O}(\alpha'^0)$, so that its contribution begins at one loop.

This assumption can be relaxed to include dilaton vectors which begin at $\mathcal{O}(\alpha'^{-1})$ and hence contribute at tree level. As noted in Section 5.2, these dilaton vectors correspond to improvement vectors $D^a = \mathcal{O}(k^0)$ in the improved VME. We have not done this computation, but we expect that the unified system remains the same in this case, except that one will now see explicitly the conjectured $\mathcal{O}(\alpha')$ terms in the eigenvalue relation (5.28) for the dilaton vector.

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Appendix A: Chiral currents in the general sigma model

In this appendix, we construct a set of non-local chiral currents in the general classical non-linear sigma model, which reduce to the currents of affine $g \times g$ in the special case of the WZW model.

The Minkowski space currents J^\pm of Section 3 satisfy the general current algebra (3.7) and the classical equations of motion

$$\partial_\mp J_a^\pm = (M_\pm)_a{}^b J_b^\pm \quad (\text{A.1a})$$

$$(M_\pm)_a{}^b = J_c^\mp (\hat{\omega}^{\pm c})_a{}^b = \partial_\mp x^i (\hat{\omega}_i^\pm)_a{}^b. \quad (\text{A.1b})$$

where $a, b = 1, \dots, \dim(M)$. These relations are the Minkowski-space analogues of the Euclidean equations of motion in (3.1). In what follows, we will also need the anticommuting exponentials $(R_\pm)_a{}^b$, which satisfy

$$\partial_\mp R_\pm + R_\pm M_\pm = 0 \quad (\text{A.2a})$$

$$R_\pm(\xi) = f_\pm(\xi^\pm) T_\mp^* e^{-\int^{\xi^\mp} d\xi'^\mp M_\pm(\xi^\pm, \xi'^\mp)} \quad (\text{A.2b})$$

$$(R_\pm)_a{}^c (R_\pm)_b{}^d G_{cd} = G_{ab} \quad (\text{A.2c})$$

where T_\mp^* is anticommuting in ξ^\mp . The matrices f_\pm are taken to be pseudoorthogonal, so that R_\pm also satisfies (A.2c).

The chiral currents \mathcal{J}^\pm of the general sigma model are then constructed as

$$\mathcal{J}_a^\pm = \pm(R_\pm)_a{}^b J_b^\pm \quad (\text{A.3a})$$

$$\partial_\mp \mathcal{J}_a^\pm = 0 \quad (\text{A.3b})$$

where the chirality condition (A.3b) follows directly from (A.1) and (A.2). We have checked that these chiral currents are Noether currents corresponding to the non-local symmetries of the sigma-model action (3.4a),

$$\delta_\pm x^i = 4\pi\alpha' \epsilon_\pm^a(\xi^\pm)(R_\pm)_a{}^i, \quad \delta S = 0 \quad (\text{A.4})$$

where $(R_\pm)_a{}^i = (R_\pm)_a{}^b e_b{}^i$ and $\epsilon_\pm(\xi_\pm)$ are the parameters of the symmetry transformations.

As a check on our construction, we consider the general chiral currents (A.3) in the special case of flat generalized connections,

$$\hat{R}_{abcd}^\pm = 0 \quad (\text{A.5})$$

which includes the case of WZW. When the generalized connections are flat, (A.2) may be solved in the form

$$\partial_i(R_\pm)_a{}^b + (R_\pm)_a{}^c(\hat{\omega}_i^\pm)_c{}^a = 0 \quad (\text{A.6})$$

so that $R_\pm(x)$ are matrix-valued functions of the coordinates $x^i(\xi)$ of the sigma model. Then we obtain the equal-time brackets

$$[J_a^+(\sigma), R_\pm(\sigma')_b{}^c] = [J_a^-(\sigma), R_\pm(\sigma')_b{}^c] = 4\pi i\alpha' \delta(\sigma - \sigma')(R_\pm)_b{}^d(\hat{\omega}_a^\pm)_d{}^c \quad (\text{A.7a})$$

$$[R_+(\sigma), R_\pm(\sigma')] = [R_-(\sigma), R_\pm(\sigma')] = 0 \quad (\text{A.7b})$$

from (A.6) and the fundamental brackets (3.5).

Using these brackets and the current algebra (3.7), the algebra of the chiral currents is computed as

$$[\mathcal{J}_a^\pm(\sigma), \mathcal{J}_b^\pm(\sigma')] = 4\pi i\alpha' \delta(\sigma - \sigma') \mathcal{F}_{ab}^\pm \mathcal{J}_c^\pm(\sigma) \pm 8\pi i\alpha' G_{ab} \partial_\sigma \delta(\sigma - \sigma') \quad (\text{A.8a})$$

$$[\mathcal{J}_a^+(\sigma), \mathcal{J}_b^-(\sigma')] = 0 \quad (\text{A.8b})$$

$$\mathcal{F}_{abc}^\pm = (R_\pm)_a{}^d (R_\pm)_b{}^e (R_\pm)_c{}^f H_{def} \quad (\text{A.8c})$$

This algebra can be identified as the bracket form of affine $g \times g$ because the structure constants \mathcal{F}_\pm in (A.8c) are indeed constants

$$\partial_i \mathcal{F}_{abc}^\pm = (R_\pm)_a{}^d (R_\pm)_b{}^e (R_\pm)_c{}^f \hat{\nabla}_i^\pm H_{def} = 0 \quad (\text{A.9})$$

when the generalized connections are flat (see (5.40)).

This result shows that all sigma model actions with $\hat{R}_{abcd}^\pm = 0$ (that is, all sigma models on parallelizable manifolds) are equivalent to the WZW action. Although this statement

is apparently well known, we have been unable to find any proof in the literature. In the standard WZW frame (2.24a), we may solve the differential equations (A.6) explicitly with the results

$$(R_+)_a{}^b = \delta_a{}^b, \quad (R_-)_a{}^b = (\Omega^{-1})_a{}^b \quad (\text{A.10a})$$

$$\mathcal{J}_a^+ = J_a^+, \quad \mathcal{J}_a^- = -(\Omega^{-1})_a{}^b J_b^- \quad (\text{A.10b})$$

$$\mathcal{F}_{ab}^{\pm c} = \frac{1}{\sqrt{\alpha'}} f_{ab}{}^c \quad (\text{A.10c})$$

where $\Omega \in \text{Aut}(g)$ is the adjoint action of g (see eqs (2.14c) and (2.14d)). Comparing with the WZW data (2.25), we see that \mathcal{J}^+ and \mathcal{J}^- are the Minkowski-space versions of the standard holomorphic and anti-holomorphic WZW currents J and \bar{J} .

It is an interesting open problem to find the algebra of the chiral currents (A.3) beyond the case of flat connections.

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